Uncertainty reduction in (static) robust optimization

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Decision dependent uncertainty in RO

- Consider the classical (static) robust optimization problem

$$\min_{\mathbf{y}\in\mathcal{Y}\subseteq\mathbb{R}_+^{n_y}} \max_{\boldsymbol{\xi}\in\Xi} \quad (\mathbf{f}+\mathbf{F}\boldsymbol{\xi})^\top \mathbf{y} \tag{RO}$$

  where $\mathcal{Y}$ is linearly constrained and $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} \mid \mathbf{D}\boldsymbol{\xi} \leq \mathbf{b}\}$ is compact.

Remark

- (RO) can be reformulated as a deterministic equivalent problem by adding a polynomial number of variables and constraints.
- (RO) does not model interactions between the decision maker and uncertain parameters.

Decision dependent uncertainty in RO

- RO with affine decision dependence is written as

$$\min_{\mathbf{x}\in\mathcal{X}\subseteq\mathbb{R}_+^{n_x},\mathbf{y}\in\mathcal{Y}\subseteq\mathbb{R}_+^{n_y}} \quad \mathbf{c}^\top\mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi^{\mathrm{AFF}}(\mathbf{x})} (\mathbf{f}+\mathbf{F}\boldsymbol{\xi})^\top \mathbf{y}$$

where $\Xi^{\mathrm{AFF}}(\mathbf{x}) = \{\boldsymbol{\xi}\in\mathbb{R}_+^{n_\xi} \mid \mathbf{D}\boldsymbol{\xi} \le \mathbf{b} + \boldsymbol{\Delta}\mathbf{x}\}$ and $\boldsymbol{\Delta}\in\mathbb{R}^{m_\xi\times n_x}$.

Remark

- This problem arises from different application contexts.
- It can also be a useful modelling tool.

Theorem (Nohadani and Sharma (2018))

Robust optimization with affine decision dependence is NP-Hard.

Uncertainty reduction in RO

- One particular affine decision dependence model is *uncertainty reduction*.
- We write (with $\mathbf{v}, \mathbf{w} \in \mathbb{R}_+^{n_\xi}$)

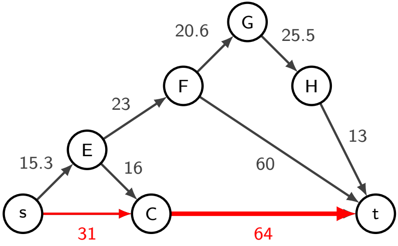
$$\Xi^{\mathrm{UR}}(\mathbf{x}) = \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} \mid \mathbf{D}\boldsymbol{\xi} \leq \mathbf{b}, \boldsymbol{\xi} \leq \mathbf{v} + \mathbf{w} \circ (\mathbb{1} - \mathbf{x})\}.$$

- Motivation: repair, investment, market studies, etc.
- Reduction is all-or-nothing, *i.e.*, $\mathbf{x} \in \{0, 1\}^{n_\xi}$.

Remark

If $v_i = 0$ then $x_i = 1$ will completely reduce ξ_i, *i.e.*, $\xi_i = 0$.

Example: shortest path problem [1]

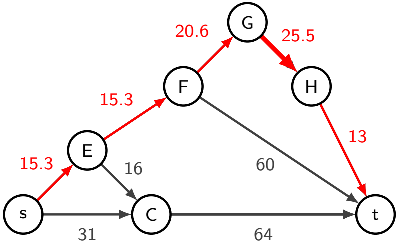


	Path	Nominal	Worst-case
Nominal	s-C-t	95	127

- $\mathcal{Y}$ contains the flow constraints.
- $f_e = \bar{f}_e(1 + 0.5\xi_e)$.
- $\Xi(\mathbf{x}) = \{\boldsymbol{\xi} \in \mathbb{R}_+^{|E|} \mid \sum_{e \in E} \xi_e \leq 1, \xi_e \leq 1 - 0.8x_e \quad \forall e \in E\}$.
- $\mathcal{X} = \{\mathbf{x} \in \{0,1\}^{|E|} \mid \sum_{e \in E} x_e \leq 1\}$.

[1] from Nohadani and Sharma (2018)

Example: shortest path problem [1]



	Path	Nominal	Worst-case
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Robust	s-E-F-G-H-t	97.4	110.15

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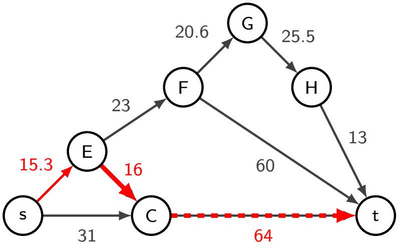
Example: shortest path problem [1]



	Path	Nominal	Worst-case
Nominal	s-C-t	95	127
Robust	s-E-F-G-H-t	97.4	110.15
UR Robust	s-E-C-t	95.3	108.1

- $\mathcal{Y}$ contains the flow constraints.
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- $\mathcal{X} = \{\mathbf{x} \in \{0,1\}^{|E|} \mid \sum_{e \in E} x_e \leq 1\}$.

[1] from Nohadani and Sharma (2018)

Decision dependence in the literature

- General models:
  - Nohadani and Sharma (2018)
  - Zeng and Wang (2022)
- As a modeling tool:
  - Spacey et al. (2012)
  - Poss (2013), Poss (2014)
  - Hanasusanto et al. (2015)

Remark

Static robust optimization problems with decision-dependent uncertainty sets have close connections to bilevel programming and generalized semi-infinite programming.

Uncertainty reduction in robust combinatorial optimization

- In this talk, we will be interested in problems of the form:

$$\min_{\mathbf{x}\in\mathcal{X}\subseteq\{0,1\}^n,\ \mathbf{y}\in\mathcal{Y}\subseteq\{0,1\}^n} \mathbf{c}^\top\mathbf{x} + \max_{\boldsymbol{\xi}\in\mathbb{R}^n_+} (\mathbf{f}+\mathbf{F}\boldsymbol{\xi})^\top\mathbf{y} \qquad \text{(UR-Min-Max)}$$

$$\text{s.t.} \quad \mathbf{d}^\top\boldsymbol{\xi} \leq b$$

$$\boldsymbol{\xi} \leq \mathbf{v} + \mathbf{w} \circ (\mathbb{1} - \mathbf{x})$$

with $\mathbf{F}$ a diagonal matrix.

Remark

We assume that uncertainty is only present in the objective function and the uncertainty set has a single "complicating" constraint for ease of exposition.

Uncertainty reduction in robust combinatorial optimization

- In this talk, we will be interested in problems of the form:

$$\min_{\mathbf{x}\in\mathcal{X}\subseteq\{0,1\}^n,\ \mathbf{y}\in\mathcal{Y}\subseteq\{0,1\}^n} \mathbf{c}^\top\mathbf{x} + \max_{\boldsymbol{\xi}\in\mathbb{R}^n_+} (\mathbf{f}+\mathbf{F}\boldsymbol{\xi})^\top\mathbf{y} \quad \text{(UR-Min-Max)}$$
$$\text{s.t.} \quad \mathbf{d}^\top\boldsymbol{\xi} \le b$$
$$\boldsymbol{\xi} \le \mathbf{v} + \mathbf{w}\circ(\mathbb{1}-\mathbf{x})$$

with **F** a diagonal matrix.

Remark

- (UR-Min-Max) is NP-Hard even when the underlying combinatorial problem is polynomially solvable.
- This is in contrast to robust combinatorial optimization problems without decision dependence.

An algorithmic approach

Proposition

(UR-Min-Max) *can be solved as at most* $n+1$ *deterministic (bilinear) problems in the* $\mathcal{X} \times \mathcal{Y}$ *space.*

Proof.

- Assuming $\mathbf{v} = \mathbf{0}$ and $\mathbf{F} = \mathbf{I}$ for ease of exposition:

$$\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + \max_{\boldsymbol{\xi}\in\mathbb{R}^n_+} \quad \boldsymbol{\xi}^\top \mathbf{y} \qquad \text{(UR-Min-Max)}$$

$$\text{s.t.} \quad \sum_{j\in[n]} d_j \xi_j \leq b \qquad (\theta)$$

$$\xi_j \leq w_j(1 - x_j) \quad \forall j \in [n] \qquad (\boldsymbol{\pi})$$

An algorithmic approach

Proof.

- Through LP duality:

$$\min_{\substack{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}\\ \theta\in\mathbb{R}_+,\boldsymbol{\pi}\in\mathbb{R}^n_+}} \mathbf{c}^\top\mathbf{x} + \mathbf{f}^\top\mathbf{y} + b\theta + \sum_{j\in[n]} w_j(1-x_j)\pi_j$$

$$\text{s.t.} \quad d_j\theta + \pi_j \geq y_j \qquad \forall j \in [n].$$

- In any optimal solution, given $\mathbf{x}, \mathbf{y}, \theta$, we have that:

$$\pi_j^* = [y_j - d_j\theta]^+ \qquad \forall j \in [n]$$

where $[a]^+ := \max\{a, 0\}$.

An algorithmic approach

Proof.

$$\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y},\theta\in\mathbb{R}_+} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + b\theta + \sum_{j\in[n]} w_j(1 - x_j)\, [y_j - d_j\theta]^+$$

- The maximum function is convex but nonlinear in $\mathbf{y}$.
- Since $\mathcal{Y} \subseteq \{0,1\}^n$, we have for $j \in [n]$:

$$y_j = 1 \implies [y_j - d_j\theta]^+ = [1 - d_j\theta]^+$$
$$y_j = 0 \implies [y_j - d_j\theta]^+ = [-d_j\theta]^+$$

- We then obtain the linear expression in $\mathbf{y}$:

$$[y_j - d_j\theta]^+ = [1 - d_j\theta]^+ y_j + [-d_j\theta]^+(1 - y_j) \qquad \forall j \in [n]$$

An algorithmic approach

Proof.

- Substituting, we obtain:

$$\min_{\substack{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}\\ \theta\in\mathbb{R}_+}} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + b\theta + \sum_{j\in[n]} w_j(1-x_j)\left([1-d_j\theta]^+ y_j + [-d_j\theta]^+(1-y_j) \right).$$

Remark

For fixed $\mathbf{x}, \mathbf{y}$, the problem can be stated as minimizing a positive-weighted combination of piecewise affine convex functions of $\theta \in \mathbb{R}_+$.

An algorithmic approach

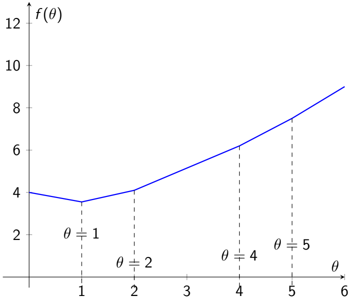

Figure: $f(\theta) = 1.5\theta + [1 - \theta]^+ + [1 - 0.5\theta]^+ + [1 - 0.25\theta]^+ + [1 - 0.2\theta]^+$

Remark

An optimal solution is obtained as one of the breakpoints of the individual functions: $\theta = \frac{1}{d_j}$ for $[1 - d_j\theta]^+$ when $d_j > 0$ and $\theta = 0$ for $[-d_j\theta]^+$.

An algorithmic approach

Proof.

- In the worst case $d_j > 0$ for $j \in [n]$.
- Therefore $\theta^* \in \{0, \frac{1}{d_1}, \dots, \frac{1}{d_n}\}$.
- (UR-Min-Max) can be solved as $n+1$ problems:

$$b\bar{\theta} + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + \sum_{j\in[n]} w_j(1-x_j)\left([1-d_j\bar{\theta}]^+ y_j + [-d_j\bar{\theta}]^+(1-y_j)\right).$$

each time fixing $\bar{\theta} = \frac{1}{d_j}$ for $j = 1, \dots, n$ with $d_j > 0$ (plus $\bar{\theta} = 0$).

An algorithmic approach

- This approach works for:
  - any $\mathbf{v} \geq 0$ and any diagonal matrix $\mathbf{F}$
  - any polyhedral uncertainty set (with multiple "complicating constraints")
  - multiple constraints affected by uncertainty (and not just the objective function)

Attention!

In the last two cases the number of deterministic problems that needs to be solved increases exponentially in the number of constraints.

So how do we solve this?

$$b\bar{\theta} + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + \sum_{j\in[n]} w_j(1 - x_j)\left([1 - d_j\bar{\theta}]^+ y_j + [-d_j\bar{\theta}]^+(1 - y_j)\right).$$

Remark

- Despite only involving **x** and **y** this problem has bilinear terms.
- The bilinear terms can be linearized using the McCormick envelope.
- We will show two cases in which this problem can be solved in polynomial time.

Solution as a combinatorial problem

Corollary

If $\mathcal{X} = \{0,1\}^n$, an optimal solution of (UR-Min-Max) *can be obtained by solving at most $n+1$ deterministic problems of the same form as:*

$$\min_{\mathbf{y}\in\mathcal{Y}} \tilde{\mathbf{f}}^\top \mathbf{y}. \qquad \text{(Combinatorial)}$$

Remark

If (Combinatorial) is polynomially solvable for all $\tilde{\mathbf{f}} \in \mathbb{R}^n$ then (UR-Min-Max) is polynomially solvable.

Solution as a combinatorial problem

Proof.

- For given $\bar{\theta}$, we need to solve:

$$b\bar{\theta} + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + \sum_{j\in[n]} w_j(1 - x_j)\left([1 - d_j\bar{\theta}]^+ y_j + [-d_j\bar{\theta}]^+(1 - y_j)\right).$$

- By rearranging the terms, we obtain:

$$K(\bar{\theta}) + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{f}^\top \mathbf{y} + \sum_{j\in[n]} w_j([1 - d_j\bar{\theta}]^+ - [-d_j\bar{\theta}]^+)y_j + \mathbf{c}^\top \mathbf{x} - \sum_{j\in[n]} w_j x_j \left([1 - d_j\bar{\theta}]^+ y_j + [-d_j\bar{\theta}]^+(1 - y_j)\right).$$

Solution as a combinatorial problem

Proof.

- We write as:

$$K(\bar{\theta}) + \min_{\mathbf{y}\in\mathcal{Y}} \Bigg\{ \mathbf{f}^\top \mathbf{y} + \sum_{j\in[n]} w_j([1-d_j\bar{\theta}]^+ - [-d_j\bar{\theta}]^+)y_j$$

$$+ \min_{\mathbf{x}\in\mathcal{X}} \mathbf{c}^\top \mathbf{x} - \sum_{j\in[n]} w_j x_j \left([1-d_j\bar{\theta}]^+ y_j + [-d_j\bar{\theta}]^+(1-y_j)\right) \Bigg\}.$$

- If $\bar{\mathbf{y}} \in \mathcal{Y}$ is fixed, the inner problem becomes:

$$\min_{\mathbf{x}\in\mathcal{X}} \; \mathbf{c}^\top \mathbf{x} - \sum_{j\in[n]} w_j x_j \left([1-d_j\bar{\theta}]^+ \bar{y}_j + [-d_j\bar{\theta}]^+(1-\bar{y}_j)\right).$$

Solution as a combinatorial problem

Proof.

- Rearranging, we obtain:

$$\min_{\mathbf{x}\in\mathcal{X}} \sum_{j\in[n]} \left(c_j - w_j\left([1-d_j\bar{\theta}]^+\bar{y}_j + [-d_j\bar{\theta}]^+(1-\bar{y}_j)\right)\right) x_j.$$

- We remark that if $\mathcal{X} = \{0,1\}^n$ then:

$$\min_{\mathbf{x}\in\{0,1\}^n} \sum_{j\in[n]} \left(c_j - w_j\left([1-d_j\bar{\theta}]^+\bar{y}_j + [-d_j\bar{\theta}]^+(1-\bar{y}_j)\right)\right) x_j$$

$$= \sum_{j\in[n]} \min_{x_j\in\{0,1\}} \left(c_j - w_j\left([1-d_j\bar{\theta}]^+\bar{y}_j + [-d_j\bar{\theta}]^+(1-\bar{y}_j)\right)\right) x_j$$

i.e., the problem decomposes over $\mathbf{x}$.

Solution as a combinatorial problem

Proof.

- We focus on the problem over each $x_j \in \{0,1\}$.

$$z_j = \min_{x_j \in \{0,1\}} \left(c_j - w_j\left([1-d_j\bar{\theta}]^+\bar{y}_j + [-d_j\bar{\theta}]^+(1-\bar{y}_j)\right)\right) x_j$$

- Since $\mathcal{Y} \subseteq \{0,1\}^n$ and $x_j \in \{0,1\}$, we have:

$$\bar{y}_j = 0 \implies z_j = [c_j - w_j[-d_j\bar{\theta}]^+]^-$$
$$\bar{y}_j = 1 \implies z_j = [c_j - w_j[1-d_j\bar{\theta}]^+]^-$$

where $[a]^- = \min\{0, a\}$.

- We then obtain the linear expression in $\bar{\mathbf{y}}$:

$$z_j = [c_j - w_j[1-d_j\bar{\theta}]^+]^-\bar{y}_j + [c_j - w_j[-d_j\bar{\theta}]^+]^-(1-\bar{y}_j)$$

Solution as a combinatorial problem

Proof.

- Putting it all together...
- Given $\bar{\theta}$, solve:

$$K'(\bar{\theta}) + \min_{\mathbf{y}\in\mathcal{Y}} \sum_{j\in[n]} \Big(f_j + w_j[1-d_j\bar{\theta}]^+ - w_j[-d_j\bar{\theta}]^+ + [c_j - w_j[1-d_j\bar{\theta}]^+]^- - [c_j - w_j[-d_j\bar{\theta}]^+]^-\Big)\, y_j$$

- In other words, solve a problem of the form:

$$\min_{\mathbf{y}\in\mathcal{Y}\subseteq\{0,1\}^n} \tilde{\mathbf{f}}^\top \mathbf{y}$$

where $\tilde{\mathbf{f}}$ is completely determined by data.

□

Solution as a linear programming problem

- $\mathcal{X} = \{\mathbf{x} \in \{0,1\}^n \mid \mathbf{A}^x \mathbf{x} \leq \mathbf{b}^x\}, \mathbf{A}^x \geq 0$
- $\mathcal{Y} = \{\mathbf{y} \in \{0,1\}^n \mid \mathbf{A}^y \mathbf{y} \leq \mathbf{b}^y\}$

Corollary

If $\mathbf{c}, \mathbf{d} \geq 0$ *and* $\mathbf{A}' := \begin{pmatrix} \mathbf{A}^x & 0 \\ 0 & \mathbf{A}^y \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}$ *is totally unimodular and* $\mathbf{b}^x, \mathbf{b}^y \in \mathbb{Z}$*, an optimal solution of* (UR-Min-Max) *can be obtained by solving at most* $n+1$ *linear programs with constraint matrix* $\mathbf{A}'$*.*

Solution as a linear programming problem

Proof.

- For given $\bar{\theta}$, we need to solve:

$$b\bar{\theta} + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} + \sum_{j\in[n]} w_j(1 - x_j)\left([1 - d_j\bar{\theta}]^+ y_j + [-d_j\bar{\theta}]^+(1 - y_j)\right).$$

- Assuming $\mathbf{d} \geq 0$, and since $\bar{\theta} \geq 0$, we obtain:

$$b\bar{\theta} + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \tilde{\mathbf{f}}(\bar{\theta})^\top \mathbf{y} - \sum_{j\in[n]} w_j[1 - d_j\bar{\theta}]^+ x_j y_j,$$

where $\tilde{f}_j(\bar{\theta}) = f_j + w_j[1 - d_j\bar{\theta}]^+$ for $j \in [n]$.

Solution as a linear programming problem

Proof.

$$b\bar{\theta} + \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \tilde{\mathbf{f}}(\bar{\theta})^\top \mathbf{y} - \sum_{j\in[n]} w_j[1 - d_j\bar{\theta}]^+ x_j y_j,$$

- Assuming $\mathbf{c}, \mathbf{A}^x \geq 0$, we have that, $\mathbf{x} \leq \mathbf{y}$ in any optimal solution.
- We may therefore write:

$$b\bar{\theta} + \min_{\substack{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y} \\ \mathbf{x}\leq\mathbf{y}}} \mathbf{c}^\top \mathbf{x} + \tilde{\mathbf{f}}(\bar{\theta})^\top \mathbf{y} - \sum_{j\in[n]} w_j[1 - d_j\bar{\theta}]^+ x_j y_j.$$

- Since $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ and $\mathbf{x} \leq \mathbf{y}$, we have:

$$x_j y_j = x_j \quad \forall j \in [n].$$

Solution as a linear programming problem

Proof.

- Introducing $\tilde{c}_j(\bar{\theta}) = c_j - w_j[1 - d_j\bar{\theta}]^+$ for each $j \in [n]$, we have:

$$
\begin{aligned}
\min \quad & \tilde{\mathbf{c}}(\bar{\theta})^\top \mathbf{x} + \tilde{\mathbf{f}}(\bar{\theta})^\top \mathbf{y} \\
\text{s.t.} \quad & \mathbf{A}^x \mathbf{x} \leq \mathbf{b}^x \\
& \mathbf{A}^y \mathbf{y} \leq \mathbf{b}^y \\
& \mathbf{x} - \mathbf{y} \leq \mathbf{0} \\
& (\mathbf{x}, \mathbf{y}) \in \{0,1\}^{2n}.
\end{aligned}
$$

□

Remark

If the stated TU assumptions are satisfied then the above integer program can be solved as a linear program.

MILP reformulations

- Let's come back to mathematical programming:

$$\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\mathbb{R}^n_+} \quad (\mathbf{f} + \mathbf{F}\boldsymbol{\xi})^\top \mathbf{y} \qquad \text{(UR-Min-Max)}$$

$$\text{s.t.} \quad \mathbf{d}^\top \boldsymbol{\xi} \leq b \qquad (\boldsymbol{\sigma})$$

$$\boldsymbol{\xi} \leq \mathbf{v} + \mathbf{w} \circ (\mathbb{1} - \mathbf{x}) \qquad (\boldsymbol{\pi})$$

- A monolithic bilinear formulation is obtained through LP duality.
- Will require linearizing the bilinear terms in $\boldsymbol{\pi}\mathbf{x}$.

Idea

Transfer the decision-dependence to the objective function of the adversarial problem.

MILP reformulations

Idea

Consider the uncertainty set:

$$\bar{\Xi}(\mathbf{x}) = \{\boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \mathbb{R}^n_+ \mid \mathbf{d}^\top(\boldsymbol{\xi}^1 + \boldsymbol{\xi}^2) \leq b, \boldsymbol{\xi}^1 \leq \mathbf{v}, \boldsymbol{\xi}^2 \leq \mathbf{w} \circ (\mathbb{1} - \mathbf{x})\}.$$

Observation

For any $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}$

$$\max_{\boldsymbol{\xi} \in \Xi^{\mathrm{UR}}(\mathbf{x})} (\mathbf{F}\boldsymbol{\xi})^\top \mathbf{y} = \max_{\boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \bar{\Xi}(\mathbf{x})} (\mathbf{F}(\boldsymbol{\xi}^1 + \boldsymbol{\xi}^2))^\top \mathbf{y}$$

Remark

In $\bar{\Xi}(\mathbf{x})$ we have that $\xi^2_j = 0$ when $x_j = 1$.

MILP reformulations

Proposition (generalized from Nohadani and Sharma (2018))

$$\max_{\boldsymbol{\xi} \in \Xi^{\mathrm{UR}}(\mathbf{x})} (\mathbf{F}\boldsymbol{\xi})^\top \mathbf{y} = \max_{\boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \bar{\Xi}(\mathbf{0})} (\mathbf{F}(\boldsymbol{\xi}^1 + \boldsymbol{\xi}^2))^\top \mathbf{y} - (\bar{\mathbf{\Pi}}\mathbf{x})^\top \boldsymbol{\xi}^2$$

where $\bar{\mathbf{\Pi}}$ *is a diagonal matrix with* $\pi_j^{\max}$ *for* $j \in [n_\xi]$ *on the diagonal.*

Proposition (generalized from Nohadani and Sharma (2018))

If $\mathbf{d} \geq 0$ *then* $\pi_j^{\max}$ *for* $j \in [n_\xi]$ *can be set to* $\max\left\{0, \max_{\mathbf{y} \in \mathcal{Y}} \ (\mathbf{F}^\top \mathbf{y})^\top e_j\right\}$.

Remark

If $\mathbf{F}$ is diagonal and $\mathcal{Y} \subseteq \{0,1\}^n$ then $\bar{\mathbf{\Pi}} = \max\{0, \mathbf{F}\}$.

MILP reformulations

- When $\mathbf{F} \geq 0$ and diagonal we obtain the reformulation:

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}\in\mathcal{Y}} \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}^1,\boldsymbol{\xi}^2\in\mathbb{R}^n_+} \quad & \mathbf{y}^\top \mathbf{F}\boldsymbol{\xi}^1 + (\mathbf{y}-\mathbf{x})^\top \mathbf{F}\boldsymbol{\xi}^2 && \text{(UR-Min-Max)} \\
\text{s.t.} \quad & \mathbf{d}^\top(\boldsymbol{\xi}^1+\boldsymbol{\xi}^2) \leq b && (\boldsymbol{\sigma}) \\
& \boldsymbol{\xi}^1 \leq \mathbf{v} && (\boldsymbol{\pi}) \\
& \boldsymbol{\xi}^2 \leq \mathbf{w} && (\boldsymbol{\mu})
\end{aligned}
$$

- A monolithic deterministic formulation is obtained through LP duality.
- No bilinear terms!

Results on the shortest path problem

$$\min_{\mathbf{x}\in\{0,1\}^{|A|},\mathbf{y}\in\mathcal{Y}} \max_{\boldsymbol{\xi}\in\Xi^{\mathrm{SP}}(\mathbf{x})} \mathbf{c}^\top \mathbf{x} + (\bar{\mathbf{f}} + \frac{1}{2}\boldsymbol{\xi}\circ\bar{\mathbf{f}})^\top \mathbf{y}$$

$$\Xi^{\mathrm{SP}}(\mathbf{x}) = \left\{ \boldsymbol{\xi} \in \mathbb{R}_+^{|A|} \mid \sum_{(i,j)\in A} \xi_{ij} \le \Gamma, \xi_{ij} \le 1 - \gamma_{ij} x_{ij} \quad \forall (i,j)\in A \right\}.$$

- We compare the algorithmic approach to the MIP reformulation based on big-M.
- We use a commercial (Gurobi) and an open source solver (HiGHS).
- We use LightGraphs.jl to solve deterministic SPs for the algorithmic approach.

Results on the shortest path problem

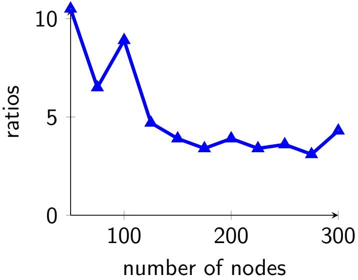

Figure: Geometric averages of the ratio between solution times using Gurobi.

- $n \in \{25, 50, \ldots, 300\}$, 10 randomly generated instances[2] in each group.
- $\bar{\mathbf{f}}$ euclidean, $\mathbf{c} = \mathbb{1}$, $\Gamma = 2$, $\gamma = 0.2$.
- Time limit of 2 hours.

[2] same generation procedure as in Nohadani and Sharma (2018)

Results on the shortest path problem

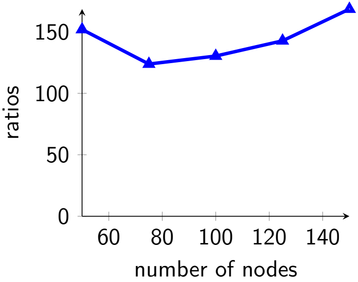

Figure: Geometric averages of the ratio between solution times using HiGHS.

- $n \in \{25, 50, \ldots, 150\}$, 10 randomly generated instances[2] in each group.
- $\bar{\mathbf{f}}$ euclidean, $\mathbf{c} = \mathbb{1}$, $\Gamma = 2$, $\gamma = 0.2$.
- Time limit of 2 hours.

[2] same generation procedure as in Nohadani and Sharma (2018)

Conclusions

take-away message

- Robust optimization with decision-dependent uncertainty sets is an interesting paradigm for many applications.
- It leads to difficult problems in general.
- Certain special cases remain polynomially solvable.

future work

- Exploring further algorithmic approaches.
- More general uncertainty-dependence structures.
- Decision-dependent adjustable robust optimization problems.

Thank you for your attention!

Arslan, Ayșe N., and Michael Poss. "Uncertainty reduction in robust optimization." Operations Research Letters (2024): 107131.