# Solving Robust Binary Optimization Problem with Budget Uncertainty

Christina Büsing, Timo Gersing, Arie Koster

Seminar Combinatorial Optimization under uncertainty through robustness

# Dicrete Optimization and MIPs



**Mixed Integer Program**

$$\min c^\top x$$
$$Ax \geq b$$
$$x \in \{0,1\}$$


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\begin{aligned}
\min\ & c^\top x \\
& Ax \geq b \\
& x \in \{0,1\}
\end{aligned}
$$


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$$\min c^\top x$$
$$Ax \geq b$$
$$x \in \{0, 1\}$$

**Historical Data/Measurements**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 506160 | 16AUG2011:1 | I1-SM/ICD | SM-2KAMME | 24AUG2012:1 | 24AUG2012:1 | 1,5824E+12 | I1 | I1LST | 5 | Im Kalender | 16AUG2011:1 | 16AUG2011 |
| 511769 | 17AUG2011:1 | I1-UNTERS | WDH | 16AUG2012:0 | 16AUG2012:0 | 2,325E+11 | I1 | I1ECHO | 5 | Im Kalender | 17AUG2011:1 | 17AUG2011 |
| 564409 | 30AUG2011:1 | I1-SM/ICD | SM-1KAMME | 28AUG2012:1 | 28AUG2012:1 | 3,7552E+12 | I1 | I1LST | 5 | Im Kalender | 30AUG2011:1 | 30AUG2011 |
| 569745 | 31AUG2011:1 | I1-SM/ICD | SM-1KAMME | 29AUG2012:1 | 29AUG2012:1 | 7,2358E+12 | I1 | I1LST | 5 | Im Kalender | 31AUG2011:1 | 31AUG2011 |
| 569745 | 31AUG2011:1 | I1-SM/ICD | SM-1KAMME | 29AUG2012:1 | 29AUG2012:1 | 7,2358E+12 | I1 | I1LST | 5 | Im Kalender | 31AUG2011:1 | 31AUG2011 |
| 569745 | 31AUG2011:1 | I1-SM/ICD | SM-1KAMME | 29AUG2012:1 | 29AUG2012:1 | 7,2358E+12 | I1 | I1LST | 5 | Im Kalender | 31AUG2011:1 | 31AUG2011 |
| 644371 | 20SEP2011:1 | I1-UNTERS | WDH | 17SEP2012:0 | 17SEP2012:0 | 3,423E+12 | I1 | IM1K | 5 | Stornierungs | 19SEP2011:0 | 20SEP2011: |
| 644372 | 20SEP2011:1 | I1-UNTERS | GKP+BGA+DI | 17SEP2012:0 | 17SEP2012:0 | 3,423E+12 | I1 | IM1K | 5 | Stornierungs | 19SEP2011:0 | 20SEP2011: |
| 648823 | 29SEP2011:1 | I1-SM/ICD | SM-2KAMME | 26SEP2012:1 | 26SEP2012:1 | 2,5708E+12 | I1 | I1LST | 5 | Im Kalender | 19SEP2011:1 | 29SEP2011: |
| 711316 | 04OCT2011:1 | I1-SM/ICD | SM-2KAMME | 25SEP2012:1 | 25SEP2012:1 | 6,3114E+11 | I1 | I1LST | 5 | Im Kalender | 04OCT2011:1 | 04OCT2011 |
| 743860 | 11OCT2011:1 | I1-SM/ICD | SM-2KAMME | 06SEP2012:1 | 06SEP2012:1 | 3,9465E+12 | I1 | I1LST | 5 | Im Kalender | 11OCT2011:1 | 11OCT2011 |
| 760872 | 14OCT2011:1 | I1-SM/ICD | SM-1KAMME | 11JUL2012:1 | 11JUL2012:1 | 2,9754E+11 | I1 | I1LST | 5 | Im Kalender | 14OCT2011:1 | 14OCT2011 |
| 533627 | 12DEC2011:0 | I1-SM/ICD | ICD-BIVENT | 10AUG2012:1 | 10AUG2012:1 | 1,4495E+10 | I1 | I1LST | 5 | Stornierungs | 23AUG2011:1 | 12DEC2011 |
| 034844 | 16DEC2011:1 | I1-SM/ICD | ICD-2KAMME | 15JUN2012:1 | 15JUN2012:1 | 1,264E+12 | I1 | I1LST | 5 | Im Kalender | 16DEC2011:1 | 16DEC2011 |
| 845266 | 03NOV2011: | I1-SM/ICD | SM-2KAMME | 27JUL2012:0 | 27JUL2012:0 | 1,2934E+12 | I1 | I1LST | 5 | Im Kalender | 03NOV2011: | 03NOV201 |
| 817375 | 27OCT2011:1 | MED1 | HOLTER3TAG | 16JUL2012:1 | 16JUL2012:1 | 3,5175E+12 | I1 | I1PO | 10 | Im Kalender | 27OCT2011:1 | 27OCT2011 |
| 817375 | 27OCT2011:1 | MED1 | HOLTER3TAG | 16JUL2012:1 | 16JUL2012:1 | 3,5175E+12 | I1 | I1PO | 10 | Status über | 27OCT2011:1 | 27OCT2011 |
| 147160 | 06FEB2012:1 | I1-SM/ICD | ICD-1KAMME | 13JUL2012:1 | 13JUL2012:1 | 5,6679E+12 | I1 | I1ECHO | 5 | Res. ROM/I1 | 13JAN2012:1 | 06FEB2012: |
| 112889 | 06JAN2012:1 | I1-ECHOKARI | TTE | 23MAY2012:0 | 23MAY2012:0 | 3,962E+11 | I1 | I1ECHO | 10 | Im Kalender | 06JAN2012:1 | 06JAN2012 |
| 112889 | 06JAN2012:1 | I1-ECHOKARI | TTE | 23MAY2012:0 | 23MAY2012:0 | 3,962E+11 | I1 | I1ECHO | 10 | Status über | 06JAN2012:1 | 06JAN2012 |
| 254641 | 06FEB2012:1 | I1-SM/ICD | ICD-2KAMME | 02AUG2012:0 | 02AUG2012:0 | 3,0771E+12 | I1 | I1LST | 5 | Im Kalender | 06FEB2012:1 | 06FEB2012: |
| 254641 | 06FEB2012:1 | I1-SM/ICD | ICD-2KAMME | 02AUG2012:0 | 02AUG2012:0 | 3,0771E+12 | I1 | I1LST | 5 | Stornierungs | 06FEB2012:1 | 06FEB2012: |

## PILOT4 from NETLIB library

- Constraint 372

$$
\begin{aligned}
a^T x \equiv\ & -15.79081 x_{826} - 8.598819 x_{827} - 1.88789 x_{828} - 1.362417 x_{829} \\
& -1.526049 x_{830} - 0.031883 x_{849} - 28.725555 x_{850} - 10.792065 x_{851} \\
& -0.19004 x_{852} - 2.757176 x_{853} - 12.290832 x_{854} + 717.562256 x_{855} \\
& -0.057865 x_{856} - 3.785417 x_{857} - 78.30661 x_{858} - 122.163055 x_{859} \\
& -6.46609 x_{860} - 0.48371 x_{861} - 0.615264 x_{862} - 1.353783 x_{863} \\
& -84.644257 x_{864} - 122.459045 x_{865} - 43.15593 x_{866} - 1.712592 x_{870} \\
& -0.401597 x_{871} + x_{880} - 0.946049 x_{898} - 0.946049 x_{916} \\
\geq\ & b \equiv 23.387405
\end{aligned}
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x^*_{826} &= 255.6112787181108 & x^*_{827} &= 6240.488912232100 \\
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- Can the coefficients be known with such a high accuracy?

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- Can the coefficients be known with such a high accuracy?
- Assume $0.1\%$-accurate approximation

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- Can the coefficients be known with such a high accuracy?
- Assume $0.1\%$-accurate approximation
- Worst violation: $450\%$

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- Worst violation: $450\%$
- Random perturbation: $(1 + \xi_j)a_j$

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- Can the coefficients be known with such a high accuracy?
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- Relative violation:

$$V = \frac{b - \tilde{a}^T x^*}{b} \times 100\%$$

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- Prob$\{V > 0\} = 0.5$
- Prob$\{V > 150\%\} = 0.18$
- Mean$(V) = 125\%$

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- Robust optimization: robust solutions remain (almost) always feasible

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- Prob$\{V > 0\} = 0.5$
- Prob$\{V > 150\%\} = 0.18$
- Mean$(V) = 125\%$

- Robust optimization: robust solutions remain (almost) always feasible
- Usually still very good objective value

### Definition (Robust Binary Programming with cost uncertainties)

Given a set of feasible solution $X = \{x \in \{0,1\}^n \mid Ax \leq b\}$. Let $\mathcal{S}$ be a set of scenarios defining cost functions $c^S : N \rightarrow \mathbb{R}$, $S \in \mathcal{S}$. A robust optimial solution $x^*$ is a feasible solution minimizing the worst case costs, i.e., solve

$$\min \left\{ \max_{S \in \mathcal{S}} \sum_{i=1}^{n} c_i^S x_i \mid Ax \leq b, x \in \{0,1\}^n \right\}$$

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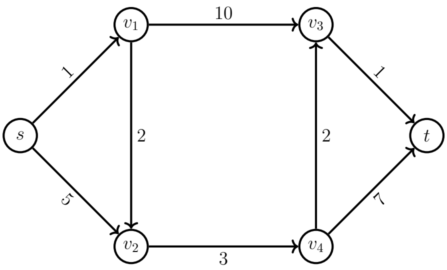
**Shortest Path Problem**



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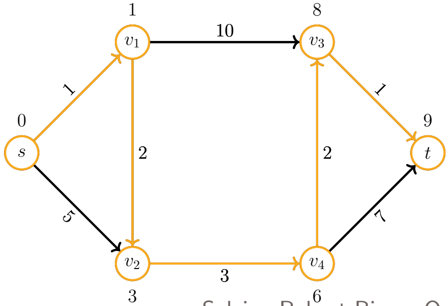
**Shortest Path Problem**



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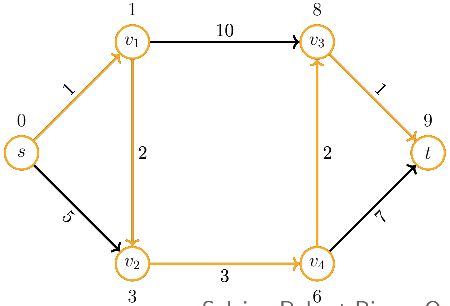


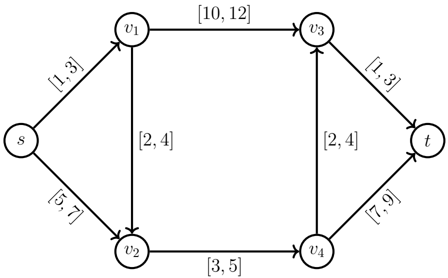

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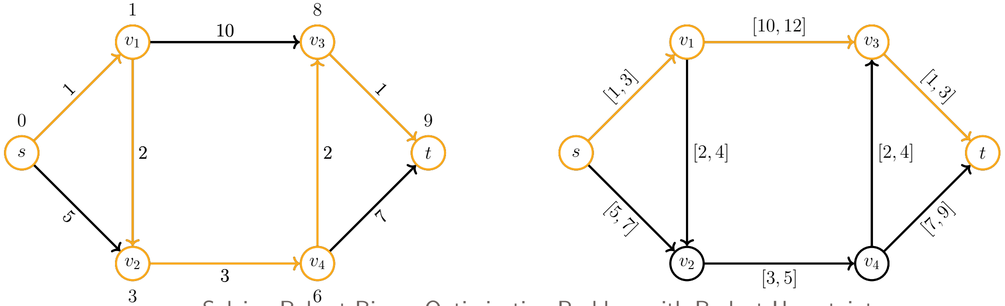

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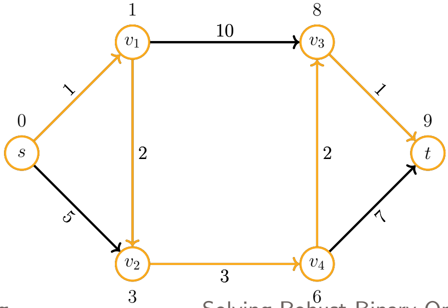


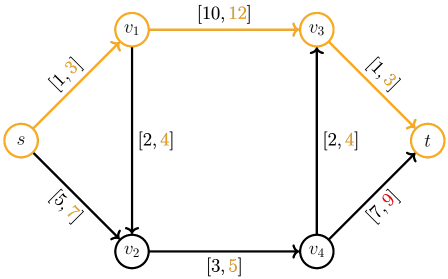

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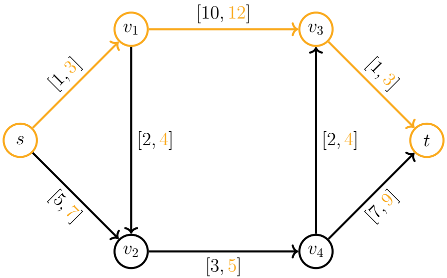

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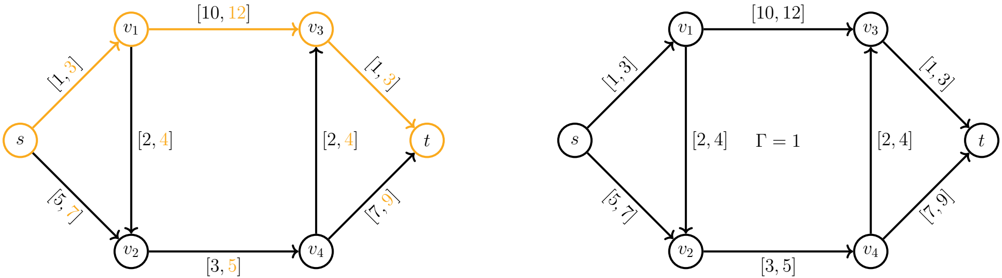

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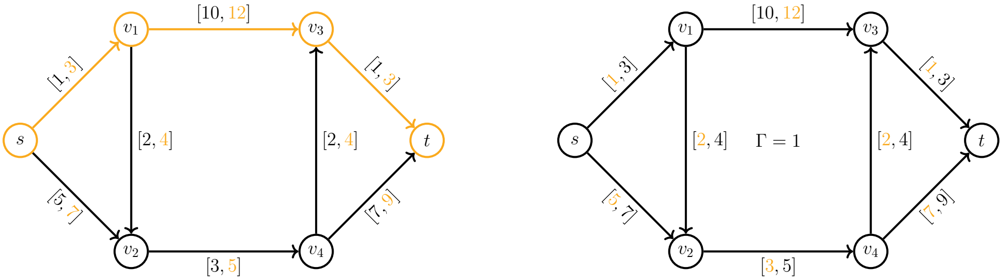

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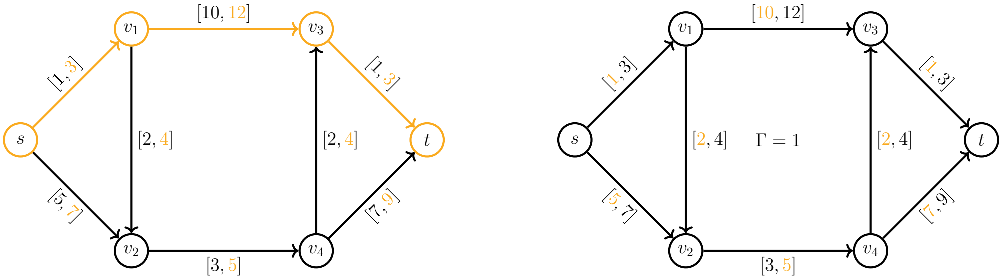

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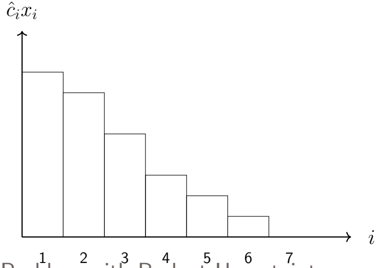

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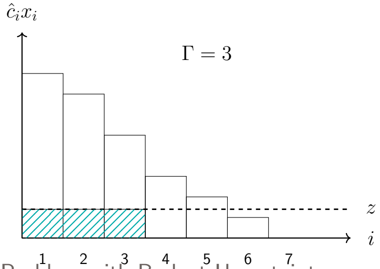

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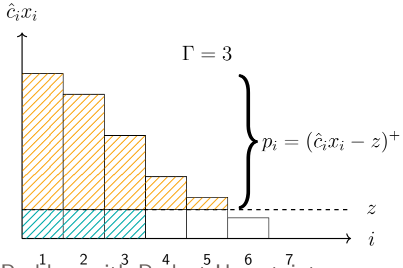

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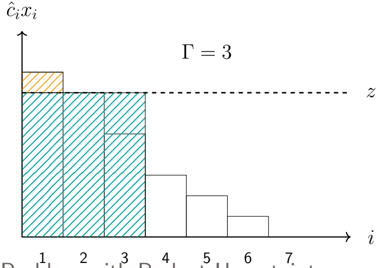

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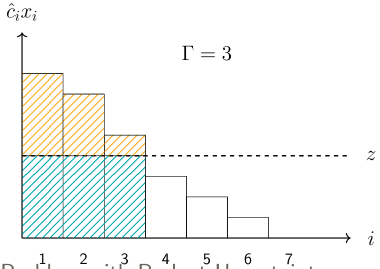

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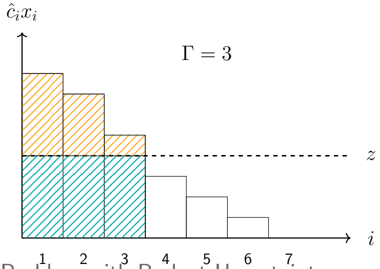

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# Practical Performance

- compact formulation, no use of big M
- let's solve some problems!
- robust knapsack:

```
              Nodes                                          Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

 23720047 15197259    11719,6862    20    11606,0000    12288,1353 55057467    5,88%
 23770901 15226459    11828,1686    23    11606,0000    12287,7700 55180967    5,87%
 23821565 15255529    12011,4041    24    11606,0000    12287,4030 55303331    5,87%
 23871269 15283781    11783,7154    22    11606,0000    12287,0473 55424214    5,87%
Elapsed time = 3434,41 sec. 5979817,73 ticks, tree = 3353,00 MB, solutions = 12
Nodefile size = 1305,88 MB 727,49 MB after compression
 23922166 15312936    12191,2914    28    11606,0000    12286,6809 55547450    5,86%
 23972875 15342043    12200,0047    28    11606,0000    12286,3216 55668913    5,86%
 24023053 15370543    11736,4889    21    11606,0000    12285,9418 55790174    5,86%
 24073375 15399019    12100,5997    25    11606,0000    12285,5821 55912495    5,86%
 24124016 15427933    12185,6295    27    11606,0000    12285,2237 56034120    5,85%
 24174475 15456520    12076,8979    25    11606,0000    12284,8592 56156283    5,85%
 24223910 15484613    11936,4399    23    11606,0000    12284,4984 56276599    5,85%
 24273972 15512958    11751,6692    24    11606,0000    12284,1408 56398552    5,84%
 24324148 15541333    11929,2290    23    11606,0000    12283,7841 56521083    5,84%
 24374451 15569768    12255,0014    27    11606,0000    12283,4225 56644319    5,84%
Elapsed time = 3528,01 sec. 6132528,45 ticks, tree = 3408,61 MB, solutions = 12
Nodefile size = 1360,88 MB 755,91 MB after compression
 24424125 15598043    11993,2112    23    11606,0000    12283,0698 56764602    5,83%
 24475158 15626928    12240,6841    27    11606,0000    12282,7013 56887548    5,83%
 24526113 15655874    12232,6254    27    11606,0000    12282,3386 57011130    5,83%
 24576245 15684253    12236,5555    30    11606,0000    12281,9808 57133348    5,82%
 24625778 15712276        cutoff          11606,0000    12281,6332 57253146    5,82%
 24676376 15740977    11992,7508    26    11606,0000    12281,2759 57374978    5,82%
 24726652 15769305    12240,2652    28    11606,0000    12280,9179 57496901    5,82%
 24777038 15797704    11615,6468    22    11606,0000    12280,5627 57618421    5,81%
 24827584 15826342    12045,1031    24    11606,0000    12280,2089 57740201    5,81%
 24877740 15854780        cutoff          11606,0000    12279,8571 57860645    5,81%
Elapsed time = 3623,59 sec. 6285119,25 ticks, tree = 3464,00 MB, solutions = 12
Nodefile size = 1416,87 MB 784,63 MB after compression
```

# Practical Performance

- compact formulation, no use of big M
- let's solve some problems!
- robust knapsack:

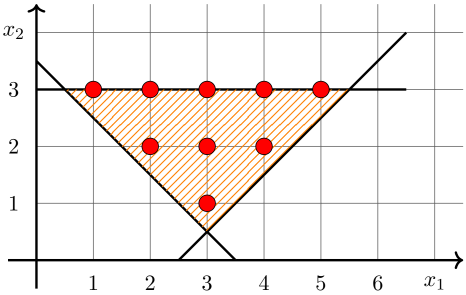
```
                  Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

 23720047 15197259    11719,6862    20    11606,0000    12288,1353 55057467    5,88%
 23770901 15226459    11828,1686    23    11606,0000    12287,7700 55180967    5,87%
 23821565 15255529    12011,4041    24    11606,0000    12287,4030 55303331    5,87%
 23871269 15283781    11783,7154    22    11606,0000    12287,0473 55424214    5,87%
Elapsed time = 3434,41 sec. 5979817,73 ticks, tree = 3353,00 MB, solutions = 12
Nodefile size = 1305,88 MB 727,49 MB after compression
 23922166 15312936    12191,2914    28    11606,0000    12286,6809 55547450    5,86%
 23972875 15342043    12200,0047    28    11606,0000    12286,3216 55668913    5,86%
 24023053 15370543    11736,4889    21    11606,0000    12285,9418 55790174    5,86%
 24073375 15399019    12100,5997    25    11606,0000    12285,5821 55912495    5,86%
 24124016 15427933    12185,6295    27    11606,0000    12285,2237 56034120    5,85%
 24174475 15456520    12076,8979    25    11606,0000    12284,8592 56156283    5,85%
 24223910 15484613    11936,4399    23    11606,0000    12284,4984 56276599    5,85%
 24273972 15512958    11751,6692    24    11606,0000    12284,1408 56398552    5,84%
 24324148 15541333    11929,2290    23    11606,0000    12283,7841 56521083    5,84%
 24374451 15569768    12255,0014    27    11606,0000    12283,4225 56644319    5,84%
Elapsed time = 3528,01 sec. 6132528,45 ticks, tree = 3408,61 MB, solutions = 12
Nodefile size = 1360,88 MB 755,91 MB after compression
 24424125 15598043    11993,2112    23    11606,0000    12283,0698 56764602    5,83%
 24475158 15626928    12240,6841    27    11606,0000    12282,7013 56887548    5,83%
 24526113 15655874    12232,6254    27    11606,0000    12282,3386 57011130    5,83%
 24576245 15684253    12236,5555    30    11606,0000    12281,9808 57133348    5,82%
 24625778 15712276        cutoff          11606,0000    12281,6332 57253146    5,82%
 24676376 15740977    11992,7508    26    11606,0000    12281,2759 57374978    5,82%
 24726652 15769305    12240,2652    28    11606,0000    12280,9179 57496901    5,82%
 24777038 15797704    11615,6468    22    11606,0000    12280,5627 57618421    5,81%
 24827584 15826342    12045,1031    24    11606,0000    12280,2089 57740201    5,81%
 24877740 15854780        cutoff          11606,0000    12279,8571 57860645    5,81%
Elapsed time = 3623,59 sec. 6285119,25 ticks, tree = 3464,00 MB, solutions = 12
Nodefile size = 1416,87 MB 784,63 MB after compression
```

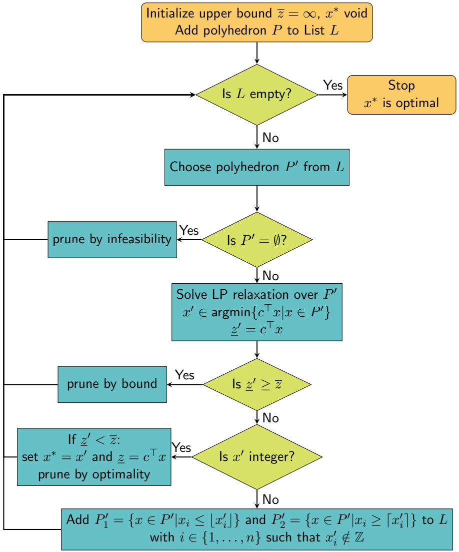
# Practical Performance

- compact formulation, no use of big M
- let's solve some problems!
- robust knapsack: 50 items can already be intractable

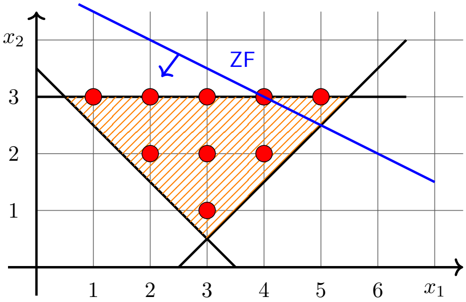
```
                Nodes                                          Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

 23720047 15197259    11719,6862    20    11606,0000    12288,1353 55057467    5,88%
 23770901 15226459    11828,1686    23    11606,0000    12287,7700 55180967    5,87%
 23821565 15255529    12011,4041    24    11606,0000    12287,4030 55303331    5,87%
 23871269 15283781    11783,7154    22    11606,0000    12287,0473 55424214    5,87%
Elapsed time = 3434,41 sec. 5979817,73 ticks, tree = 3353,00 MB, solutions = 12
Nodefile size = 1305,88 MB 727,49 MB after compression
 23922166 15312936    12191,2914    28    11606,0000    12286,6809 55547450    5,86%
 23972875 15342043    12200,0047    28    11606,0000    12286,3216 55668913    5,86%
 24023053 15370543    11736,4889    21    11606,0000    12285,9418 55790174    5,86%
 24073375 15399019    12100,5997    25    11606,0000    12285,5821 55912495    5,86%
 24124016 15427933    12185,6295    27    11606,0000    12285,2237 56034120    5,85%
 24174475 15456520    12076,8979    25    11606,0000    12284,8592 56156283    5,85%
 24223910 15484613    11936,4399    23    11606,0000    12284,4984 56276599    5,85%
 24273972 15512958    11751,6692    24    11606,0000    12284,1408 56398552    5,84%
 24324148 15541333    11929,2290    23    11606,0000    12283,7841 56521083    5,84%
 24374451 15569768    12255,0014    27    11606,0000    12283,4225 56644319    5,84%
Elapsed time = 3528,01 sec. 6132528,45 ticks, tree = 3408,61 MB, solutions = 12
Nodefile size = 1360,88 MB 755,91 MB after compression
 24424125 15598043    11993,2112    23    11606,0000    12283,0698 56764602    5,83%
 24475158 15626928    12240,6841    27    11606,0000    12282,7013 56887548    5,83%
 24526113 15655874    12232,6254    27    11606,0000    12282,3386 57011130    5,83%
 24576245 15684253    12236,5555    30    11606,0000    12281,9808 57133348    5,82%
 24625778 15712276        cutoff          11606,0000    12281,6332 57253146    5,82%
 24676376 15740977    11992,7508    26    11606,0000    12281,2759 57374978    5,82%
 24726652 15769305    12240,2652    28    11606,0000    12280,9179 57496901    5,82%
 24777038 15797704    11615,6468    22    11606,0000    12280,5627 57618421    5,81%
 24827584 15826342    12045,1031    24    11606,0000    12280,2089 57740201    5,81%
 24877740 15854780        cutoff          11606,0000    12279,8571 57860645    5,81%
Elapsed time = 3623,59 sec. 6285119,25 ticks, tree = 3464,00 MB, solutions = 12
Nodefile size = 1416,87 MB 784,63 MB after compression
```

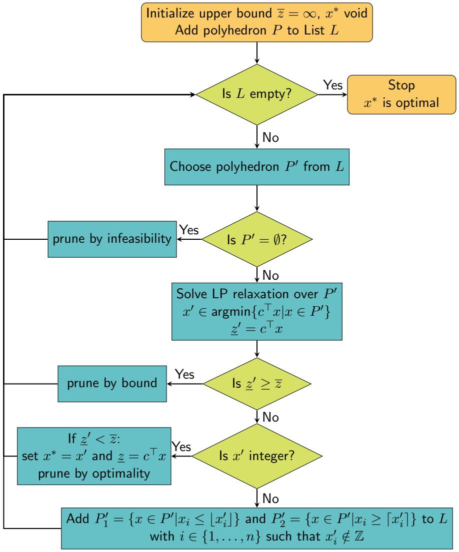
# Branch & Bound

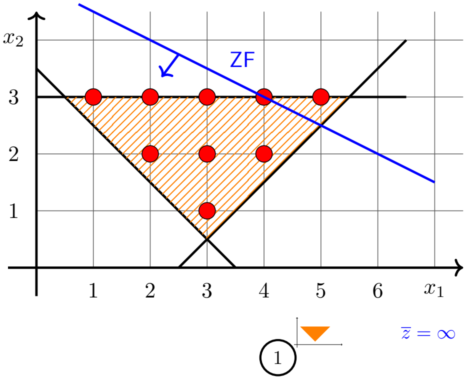


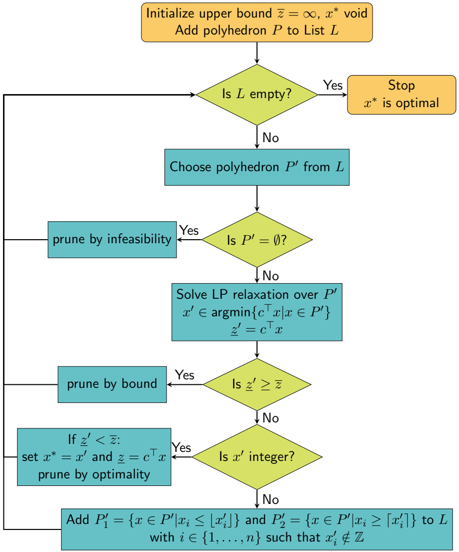



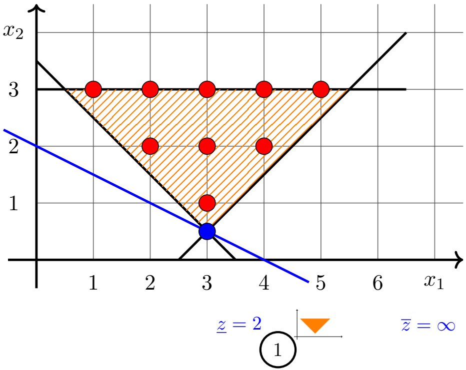
# Branch & Bound

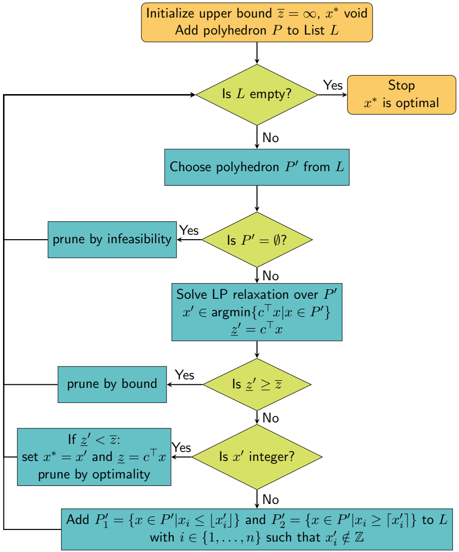


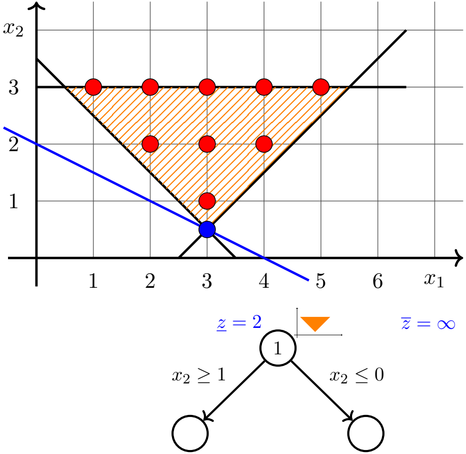

$x_2$
ZF
3
2
1
1
2
3
4
5
6
$x_1$
1
$\overline{z} = \infty$

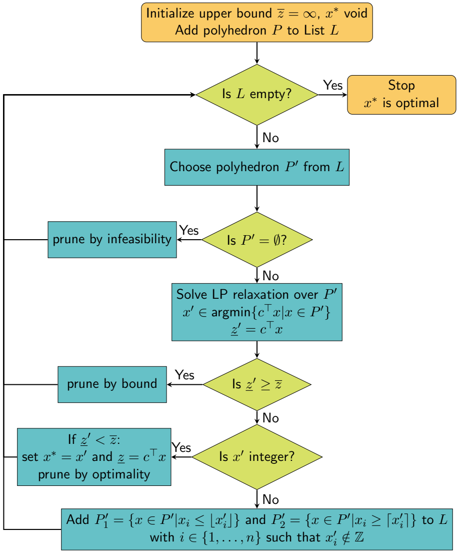

Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
prune by infeasibility
Yes
Is $P' = \emptyset$?
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
prune by bound
Yes
Is $\underline{z}' \geq \overline{z}$
No
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
Yes
Is $x'$ integer?
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x_i' \notin \mathbb{Z}$

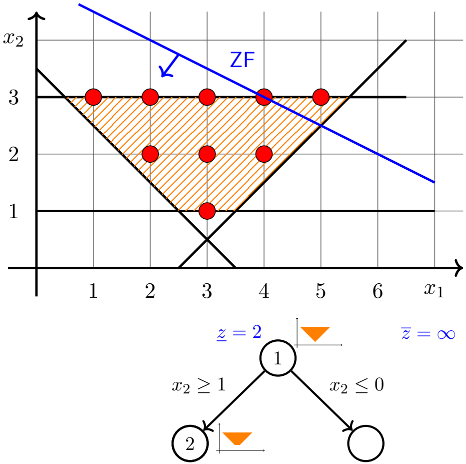

$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
$\underline{z} = 2$
1
$\overline{z} = \infty$

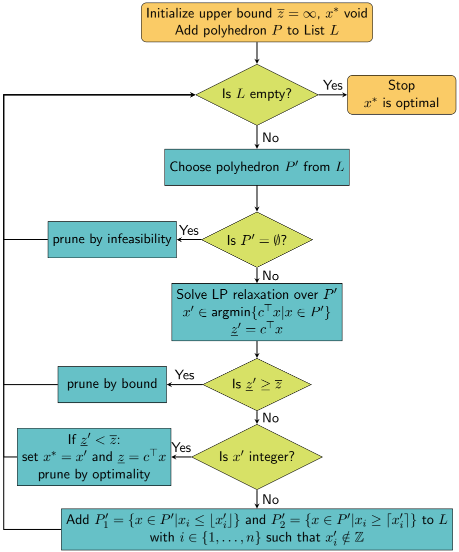

Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
Is $P' = \emptyset$?
Yes
prune by infeasibility
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
Is $\underline{z}' \geq \overline{z}$
Yes
prune by bound
No
Is $x'$ integer?
Yes
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
No
Add $P'_1 = \{x \in P' | x_i \leq \lfloor x'_i \rfloor\}$ and $P'_2 = \{x \in P' | x_i \geq \lceil x'_i \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x'_i \notin \mathbb{Z}$

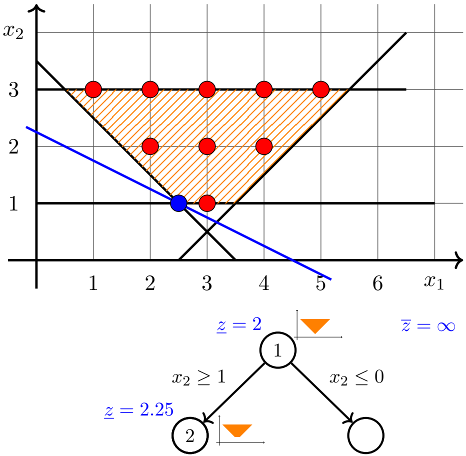
# Branch & Bound

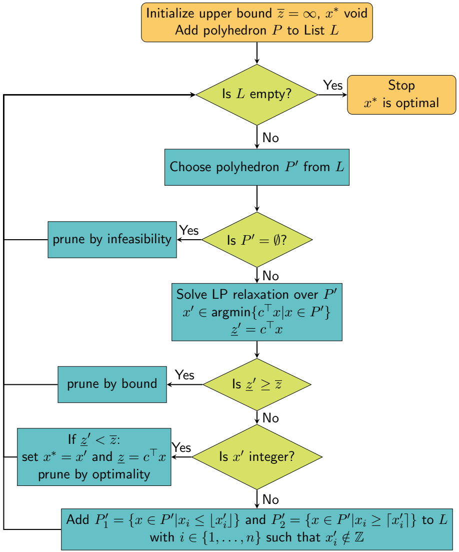


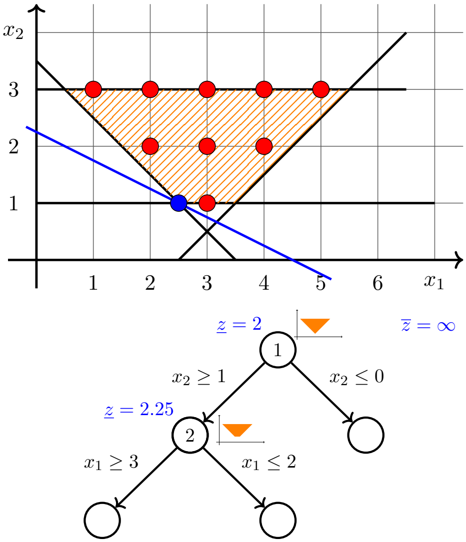



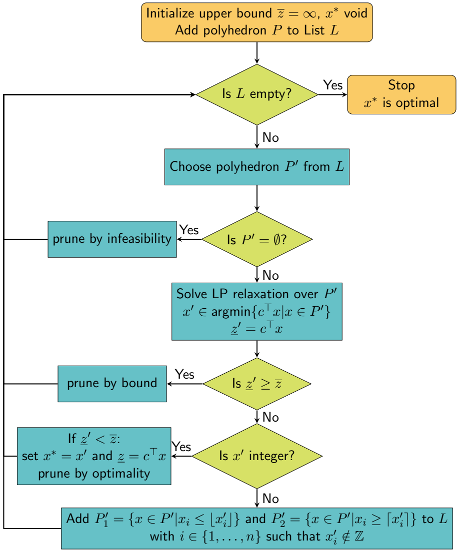

$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
ZF
$\underline{z} = 2$
$\overline{z} = \infty$
1
$x_2 \geq 1$
$x_2 \leq 0$
2

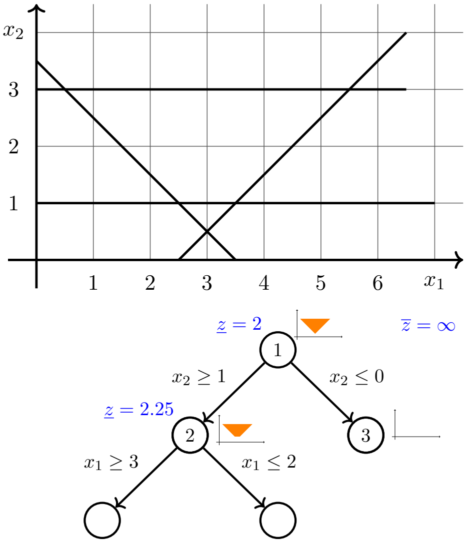

Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
Is $P' = \emptyset$?
Yes
prune by infeasibility
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
Is $\underline{z}' \geq \overline{z}$
Yes
prune by bound
No
Is $x'$ integer?
Yes
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x_i' \notin \mathbb{Z}$

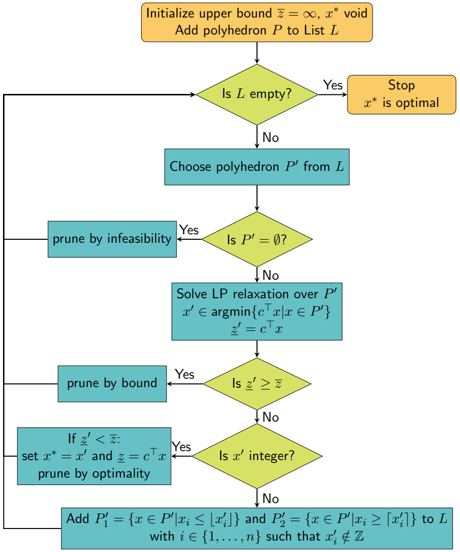

$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
$\underline{z} = 2$
$\overline{z} = \infty$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2

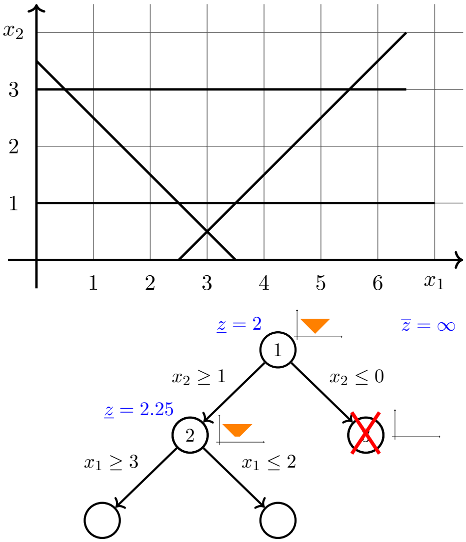

Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
prune by infeasibility
Yes
Is $P' = \emptyset$?
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
prune by bound
Yes
Is $\underline{z}' \geq \overline{z}$
No
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
Yes
Is $x'$ integer?
No
Add $P'_1 = \{x \in P' | x_i \leq \lfloor x'_i \rfloor\}$ and $P'_2 = \{x \in P' | x_i \geq \lceil x'_i \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x'_i \notin \mathbb{Z}$


$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
$\underline{z} = 2$
$\overline{z} = \infty$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2
$x_1 \geq 3$
$x_1 \leq 2$

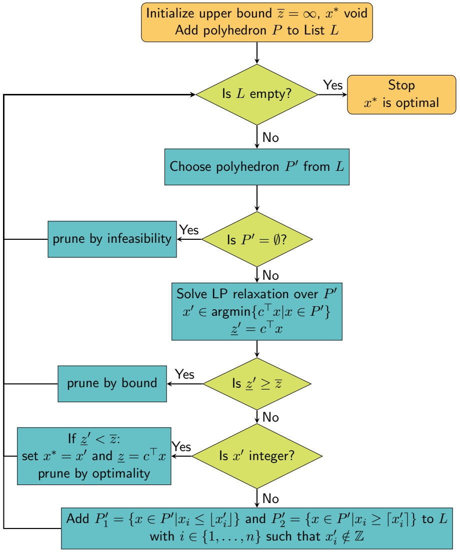

Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
prune by infeasibility
Yes
Is $P' = \emptyset$?
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
prune by bound
Yes
Is $\underline{z}' \geq \overline{z}$
No
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
Yes
Is $x'$ integer?
No
Add $P'_1 = \{x \in P' | x_i \leq \lfloor x'_i \rfloor\}$ and $P'_2 = \{x \in P' | x_i \geq \lceil x'_i \rceil\}$ to $L$
with $i \in \{1, \dots, n\}$ such that $x'_i \notin \mathbb{Z}$


$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
$\underline{z} = 2$
$\overline{z} = \infty$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2
3
$x_1 \geq 3$
$x_1 \leq 2$

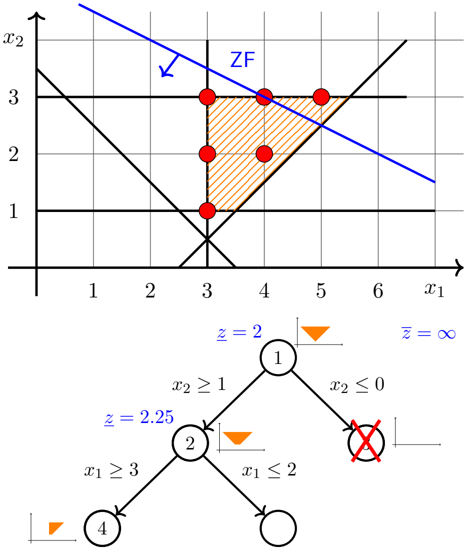

Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
prune by infeasibility
Yes
Is $P' = \emptyset$?
No
Solve LP relaxation over $P'$
$x' \in$ argmin$\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
prune by bound
Yes
Is $\underline{z}' \geq \overline{z}$
No
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
Yes
Is $x'$ integer?
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x_i' \notin \mathbb{Z}$



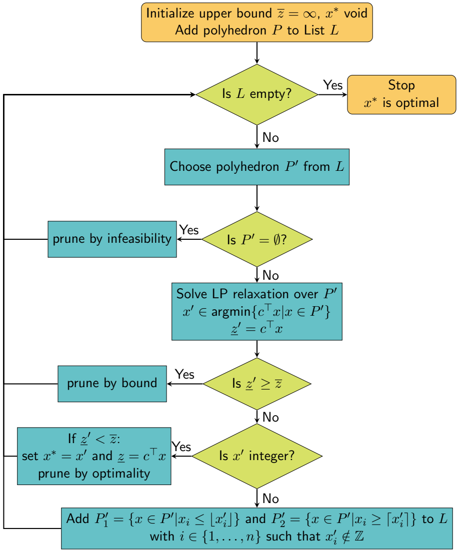
$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
$\underline{z} = 2$
$\overline{z} = \infty$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2
$x_1 \geq 3$
$x_1 \leq 2$

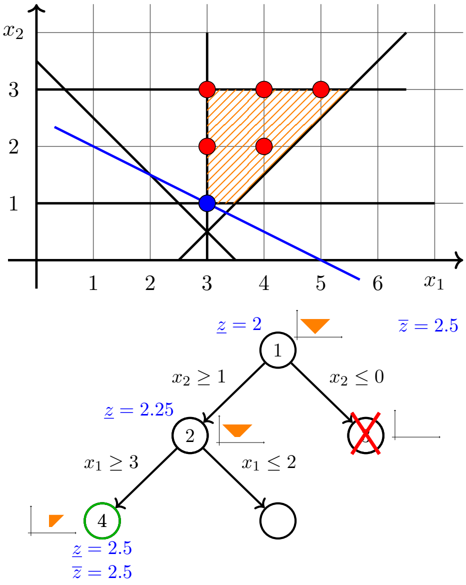
Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
Is $P' = \emptyset$?
Yes
prune by infeasibility
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
Is $\underline{z}' \geq \overline{z}$
Yes
prune by bound
No
Is $x'$ integer?
Yes
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \dots, n\}$ such that $x_i' \notin \mathbb{Z}$



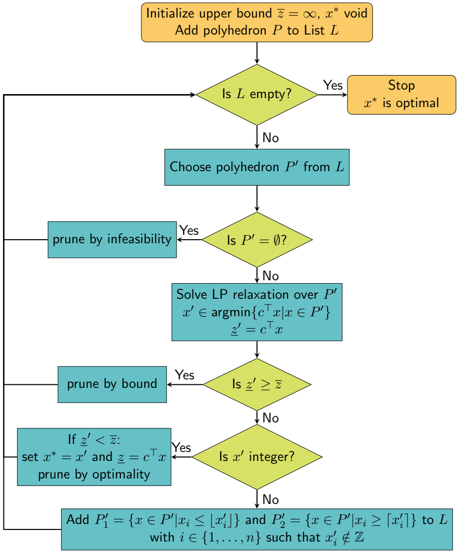
$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
ZF
$\underline{z} = 2$
$\overline{z} = \infty$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2
$x_1 \geq 3$
$x_1 \leq 2$
4

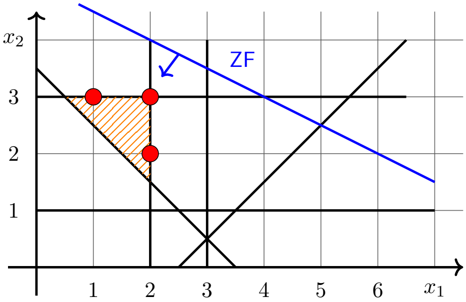
Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
Is $P' = \emptyset$?
Yes
prune by infeasibility
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
Is $\underline{z}' \geq \overline{z}$
Yes
prune by bound
No
Is $x'$ integer?
Yes
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x_i' \notin \mathbb{Z}$



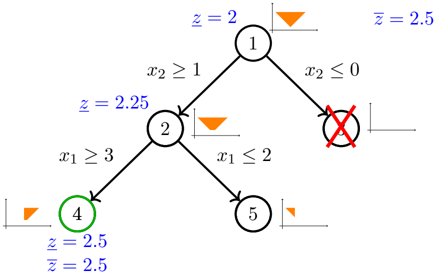
$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
$\underline{z} = 2$
$\overline{z} = 2.5$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2
$x_1 \geq 3$
$x_1 \leq 2$
4
$\underline{z} = 2.5$
$\overline{z} = 2.5$

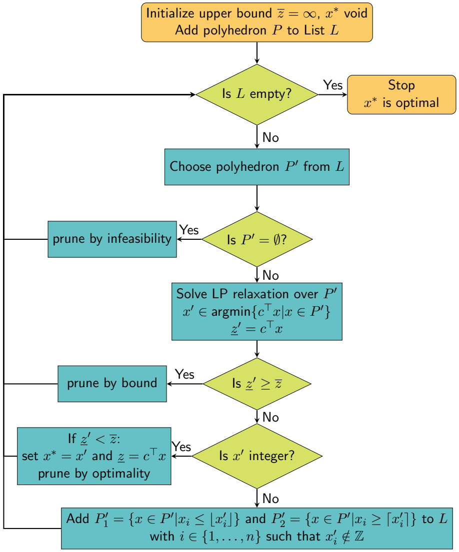
Initialize upper bound $\overline{z} = \infty$, $x^*$ void
Add polyhedron $P$ to List $L$
Is $L$ empty?
Yes
Stop
$x^*$ is optimal
No
Choose polyhedron $P'$ from $L$
prune by infeasibility
Yes
Is $P' = \emptyset$?
No
Solve LP relaxation over $P'$
$x' \in \text{argmin}\{c^\top x | x \in P'\}$
$\underline{z}' = c^\top x$
prune by bound
Yes
Is $\underline{z}' \geq \overline{z}$
No
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
Yes
Is $x'$ integer?
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x_i' \notin \mathbb{Z}$



$x_2$
3
2
1
1
2
3
4
5
6
$x_1$
ZF

$\underline{z} = 2$
$\overline{z} = 2.5$
1
$x_2 \geq 1$
$x_2 \leq 0$
$\underline{z} = 2.25$
2
3
$x_1 \geq 3$
$x_1 \leq 2$
4
5
$\underline{z} = 2.5$
$\overline{z} = 2.5$

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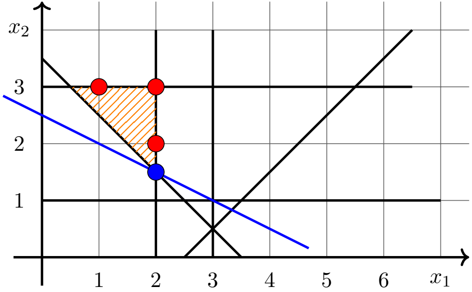
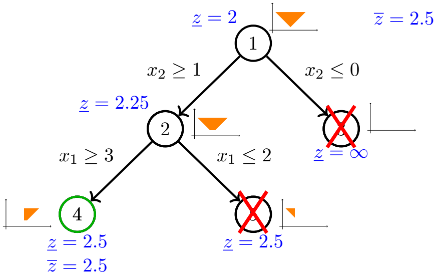
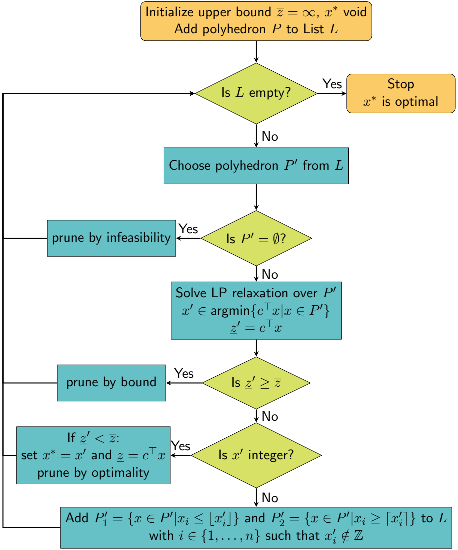
$\underline{z}' = c^\top x$
prune by bound
Yes
Is $\underline{z}' \geq \overline{z}$
No
If $\underline{z}' < \overline{z}$:
set $x^* = x'$ and $\underline{z} = c^\top x$
prune by optimality
Yes
Is $x'$ integer?
No
Add $P_1' = \{x \in P' | x_i \leq \lfloor x_i' \rfloor\}$ and $P_2' = \{x \in P' | x_i \geq \lceil x_i' \rceil\}$ to $L$
with $i \in \{1, \ldots, n\}$ such that $x_i' \notin \mathbb{Z}$

# Branch & Bound



# Bad Relaxation

- Robust model performs poorly due to weak LP-relaxation

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- **Example:** choose element with smallest weight

$$\begin{aligned} \min \quad & \sum_{i \in N} c_i x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & x \in \{0,1\}^n \end{aligned}$$

- Robust model performs poorly due to weak LP-relaxation
- **Example:** choose element with smallest weight for uniform weights

$$\begin{aligned} \min \quad & \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & x \in \{0,1\}^n \end{aligned}$$

- Robust model performs poorly due to weak LP-relaxation
- **Example:** choose element with smallest weight for uniform weights
- Consider uniform deviations, $\Gamma = 1$

$$\begin{aligned} \min \quad & \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & x \in \{0,1\}^n \end{aligned}$$

- Robust model performs poorly due to weak LP-relaxation
- **Example:** choose element with smallest weight for uniform weights
- Consider uniform deviations, $\Gamma = 1$

$$
\begin{aligned}
\min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\
\text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\
& z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

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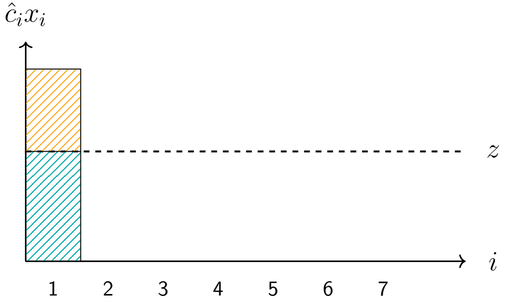
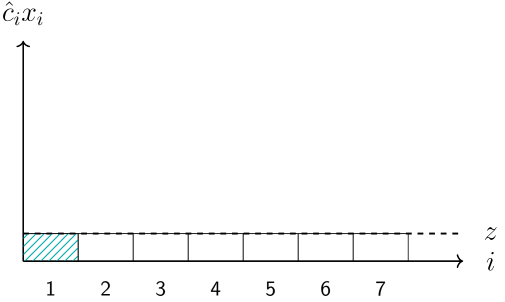
- $(1, 0, \ldots, 0)$ integer optimal with solution value $c + \hat{c}$



- Robust model performs poorly due to weak LP-relaxation
- **Example:** choose element with smallest weight for uniform weights
- Consider uniform deviations, $\Gamma = 1$

$$
\begin{aligned}
\min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c \, x_i \\
\text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\
& z + p_i \geq \hat{c} \, x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

- $(1, 0, \ldots, 0)$ integer optimal with solution value $c + \hat{c}$
- $(\frac{1}{n}, \ldots, \frac{1}{n})$ continuous optimal with solution value $c + \frac{\hat{c}}{n}$



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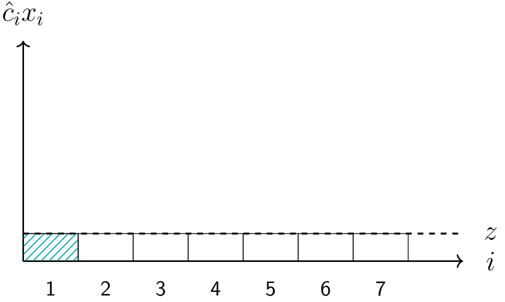
- **Example:** choose element with smallest weight for uniform weights
- Consider uniform deviations, $\Gamma = 1$

$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \end{aligned}$$

- $(1, 0, \ldots, 0)$ integer optimal with solution value $c + \hat{c}$
- $(\frac{1}{n}, \ldots, \frac{1}{n})$ continuous optimal with solution value $c + \frac{\hat{c}}{n}$
- Influence of uncertainty vanishes



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& z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\
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\end{aligned}
$$

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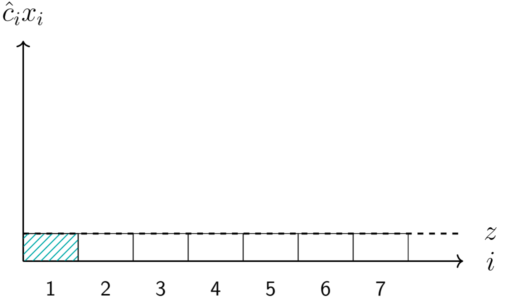



- Inherent "problem" to robust optimization: diversification

Problem

$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \end{aligned}$$

# Bad Relaxation and B&B


Node 1

$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \end{aligned}$$

$$\Rightarrow x^* = \left(\tfrac{1}{n}, \ldots, \tfrac{1}{n}\right), z^* = \tfrac{\hat{c}}{n}$$

$\overline{z} = \infty$

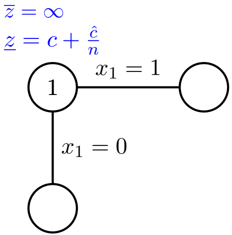
$\underline{z} = c + \frac{\hat{c}}{n}$

1

Node 1

$$
\begin{aligned}
\min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\
\text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\
& z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

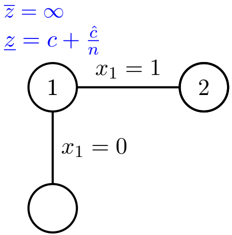
$$\Rightarrow x^* = (\tfrac{1}{n}, \ldots, \tfrac{1}{n}),\ z^* = \frac{\hat{c}}{n}$$




Node 1

$$
\begin{aligned}
\min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\
\text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\
& z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

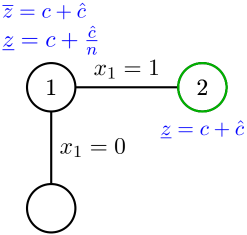
$$\Rightarrow x^* = (\tfrac{1}{n}, \ldots, \tfrac{1}{n}),\ z^* = \frac{\hat{c}}{n}$$




Node 2

$$
\begin{aligned}
\min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\
\text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\
& z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

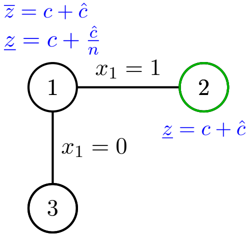
$$x_1 = 1$$




Node 2

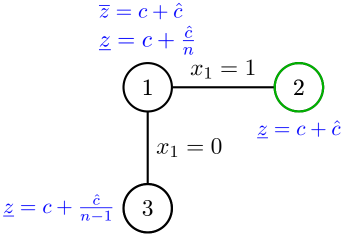
$$
\begin{aligned}
\min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\
\text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\
& z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \\
& x_1 = 1
\end{aligned}
$$

$$\Rightarrow x^* = (1, 0, ..., 0), z^* = \hat{c}$$




Node 3

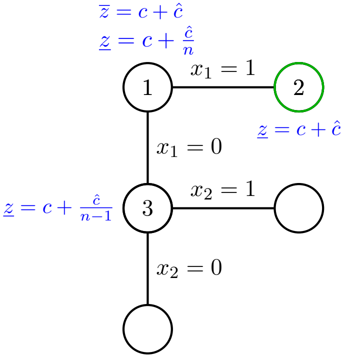
$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \\ & x_1 = 0 \end{aligned}$$




Node 3

$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \\ & x_1 = 0 \end{aligned}$$

$$\Rightarrow x^* = (0, \tfrac{1}{n-1}, \ldots, \tfrac{1}{n-1}), z^* = \tfrac{\hat{c}}{n-1}$$

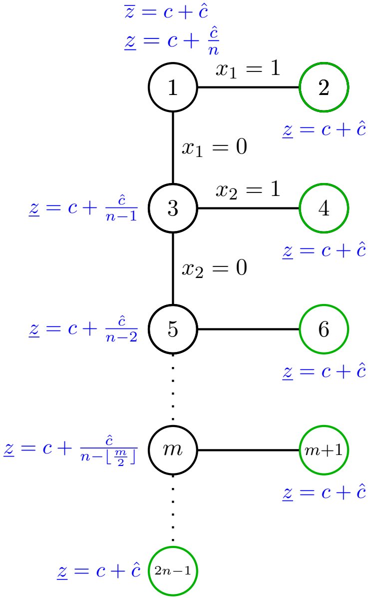



Node 3

$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \end{aligned}$$

$$x_1 = 0$$

$$\Rightarrow x^* = (0, \tfrac{1}{n-1}, \dots, \tfrac{1}{n-1}), z^* = \frac{\hat{c}}{n-1}$$




Node $2n-1$

$$\begin{aligned} \min \quad & z + \sum_{i \in N} p_i + \sum_{i \in N} c\, x_i \\ \text{s.t.} \quad & \sum_{i \in N} x_i = 1 \\ & z + p_i \geq \hat{c}\, x_i \quad \forall i \in N \\ & p \in \mathbb{R}^n, z \geq 0, x \in \{0,1\}^n \end{aligned}$$

$$x_1 = 0, x_2 = 0, \ldots, x_{n-1} = 0$$

$$\begin{aligned}
\min \quad & \Gamma z + \sum_{i \in N} p_i + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & Ax \geq b \\
& z + p_i \geq \hat{c}_i x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n_{\geq 0}, z \geq 0, x \in \{0,1\}^n
\end{aligned}$$

$$
\begin{aligned}
\min \quad & \Gamma z + \sum_{i \in N} p_i + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & Ax \geq b \\
& z + p_i \geq \hat{c}_i x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n_{\geq 0}, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

**Strong Formulations**

- Atamtürk:
  four strong versions

$$
\begin{aligned}
\min \quad & \Gamma z + \sum_{i \in N} p_i + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & Ax \geq b \\
& z + p_i \geq \hat{c}_i x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n_{\geq 0}, z \geq 0, x \in \{0,1\}^n
\end{aligned}
$$

**Strong Formulations**

- Atamtürk:
  four strong versions

**Branch on $z$**

- Bertsimas & Sim:
  $n+1$-subproblems
- Hansknecht et. al:
  Devide and Conquer

- Atamtürk Formulations: If the nominal formulation is $\alpha$-tight then the strongest formulation is also $\alpha$-tight for the robust problem [At2006]

# A Bilinear Formulation

- Atamtürk Formulations: If the nominal formulation is $\alpha$-tight then the strongest formulation is also $\alpha$-tight for the robust problem [At2006]
- Relatively small $z$ are sufficient to fulfill $p_i + z \geq \hat{c}_i x_i$ for fractional $x_i$

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- Relatively small $z$ are sufficient to fulfill $p_i + z \geq \hat{c}_i x_i$ for fractional $x_i$
- Remedy: multiply $z$ with $x_i$ to strengthen the constraint:

$$\begin{aligned}
\min \quad & \sum_{i \in N} c_i x_i + \Gamma z + \sum_{i \in N} p_i \\
\text{s.t.} \quad & Ax \geq b \\
& p_i + x_i z \geq \hat{c}_i x_i && \forall i \in N \\
& x \in \{0,1\}^n, p \in \mathbb{R}^n_{\geq 0}, z \geq 0
\end{aligned}$$

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$$
\begin{aligned}
\min \quad & \sum_{i \in N} c_i x_i + \Gamma z + \sum_{i \in N} p_i \\
\text{s.t.} \quad & Ax \geq b \\
& p_i + x_i z \geq \hat{c}_i x_i \qquad \forall i \in N \\
& x \in \{0,1\}^n, p \in \mathbb{R}^n_{\geq 0}, z \geq 0
\end{aligned}
$$

- Bilinear constraint is equivalent to $\begin{cases} p_i \geq 0 \text{ for } x_i = 0 \\ p_i + z \geq \hat{c}_i x_i \text{ for } x_i = 1 \end{cases}$

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$$
\begin{aligned}
\min \quad & \sum_{i \in N} c_i x_i + \Gamma z + \sum_{i \in N} p_i \\
\text{s.t.} \quad & Ax \geq b \\
& p_i + x_i z \geq \hat{c}_i x_i \qquad \forall i \in N \\
& x \in \{0,1\}^n, p \in \mathbb{R}^n_{\geq 0}, z \geq 0
\end{aligned}
$$

- Bilinear constraint is equivalent to $\begin{cases} p_i \geq 0 \text{ for } x_i = 0 \\ p_i + z \geq \hat{c}_i x_i \text{ for } x_i = 1 \end{cases}$

**Theorem**

*The above bilinear formulation is stronger than any polyhedral formulation.*

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$$\begin{aligned} \min \quad & \sum_{i\in N} c_i x_i + \Gamma z + \sum_{i \in N} p_i \\ \text{s.t.} \quad & Ax \geq b \\ & p_i + x_i z \geq \hat{c}_i x_i \qquad \forall i \in N \\ & x \in \{0,1\}^n, p \in \mathbb{R}^n_{\geq 0}, z \geq 0 \end{aligned}$$

- Bilinear constraint is equivalent to $\begin{cases} p_i \geq 0 \text{ for } x_i = 0 \\ p_i + z \geq \hat{c}_i x_i \text{ for } x_i = 1 \end{cases}$

**Theorem**

*The above bilinear formulation is stronger than any polyhedral formulation.*

- The bilinear formulation is impractical but the foundation for two new approaches

- For fixed $z = z'$ it holds
  $p_i = (\hat{c}_i x_i - z')^+ = (\hat{c}_i - z')^+ x_i$

$$
\begin{aligned}
\min \quad & \Gamma z + \sum_{i \in N} p_i + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & Ax \geq b \\
& z + p_i \geq \hat{c}_i x_i \quad \forall i \in N \\
& p \in \mathbb{R}^n_{\geq 0}, z = z', x \in \{0,1\}^n
\end{aligned}
$$

- For fixed $z = z'$ it holds
  $p_i = (\hat{c}_i x_i - z')^+ = (\hat{c}_i - z')^+ x_i$
- Fixing $z$ yields a nominal problem

$$\begin{aligned} \min \quad & \Gamma z' + \sum_{i \in N} (c_i + (\hat{c}_i - z')^+) x_i \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \{0,1\}^n \end{aligned}$$

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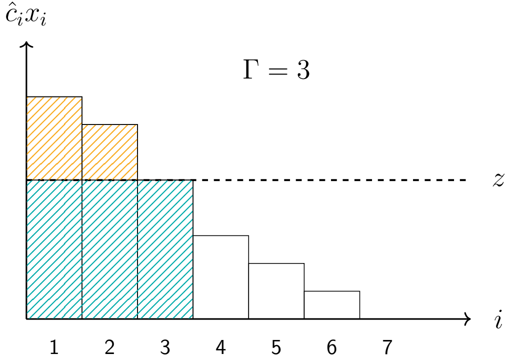

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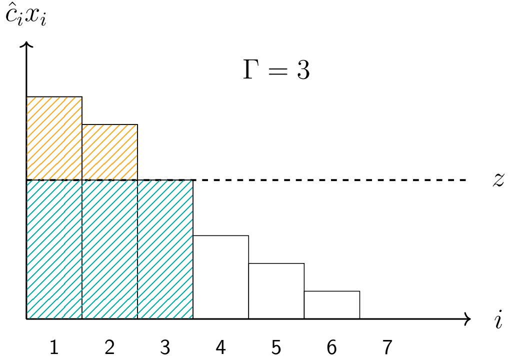

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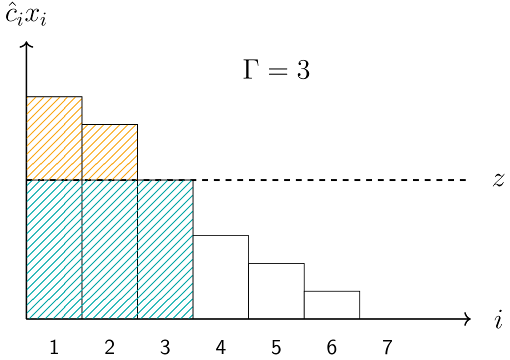

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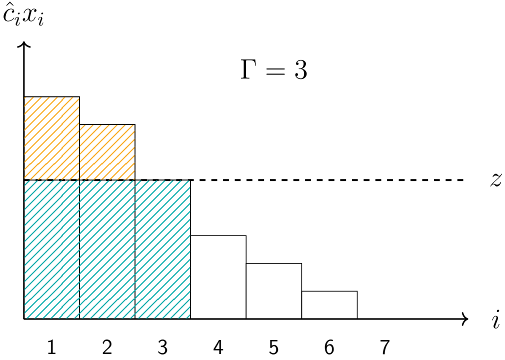

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- Prune $z$ on the fly using relations between objective values [HRS2018]

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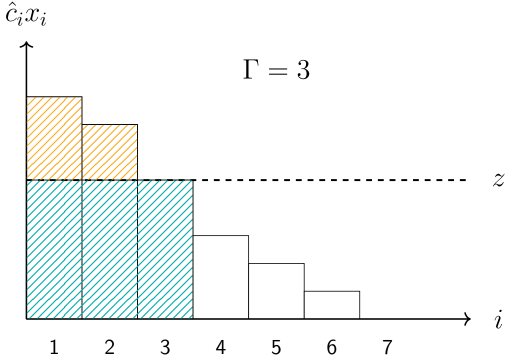

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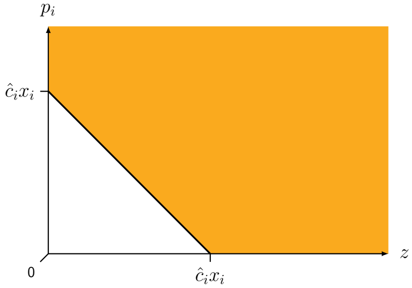


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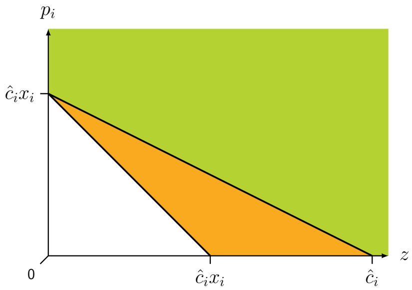


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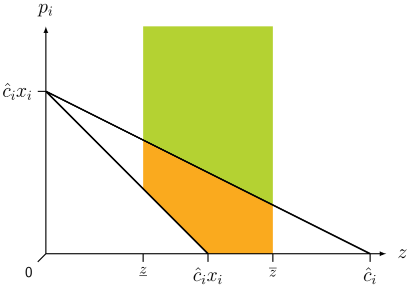


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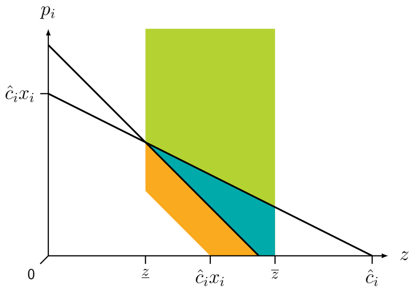


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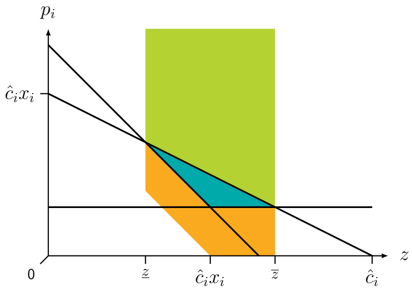


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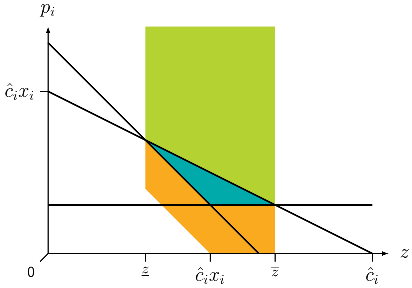


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**Proposition**

Inequalities (1) and (2) approximate the bilinear one and are equally strong if $z \in \{\underline{z}, \overline{z}\}$.

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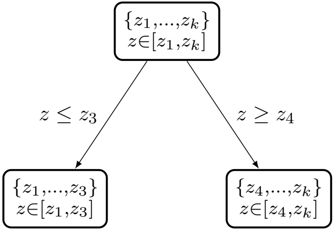

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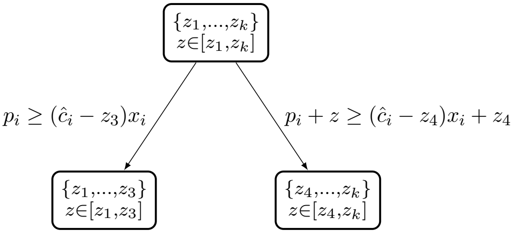

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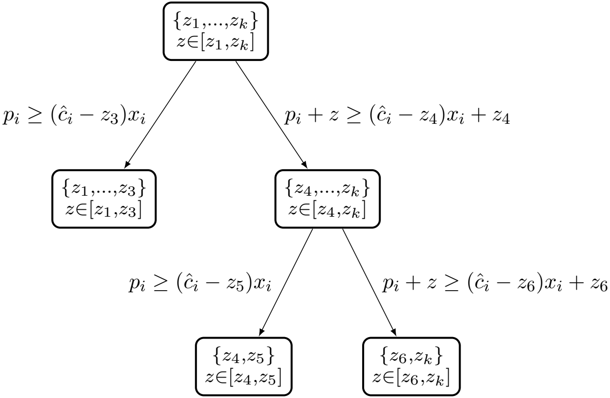

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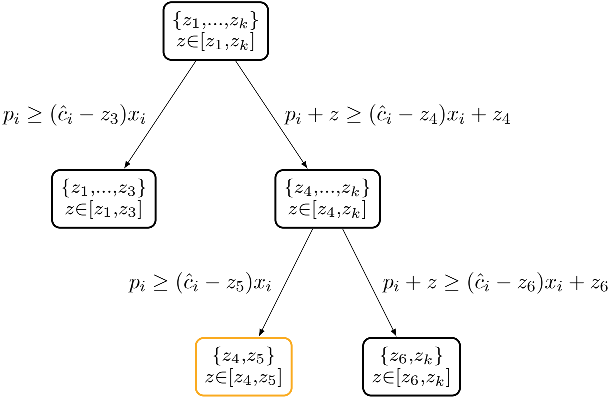

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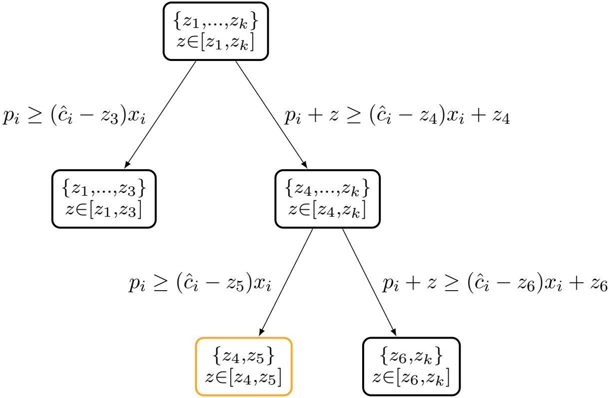

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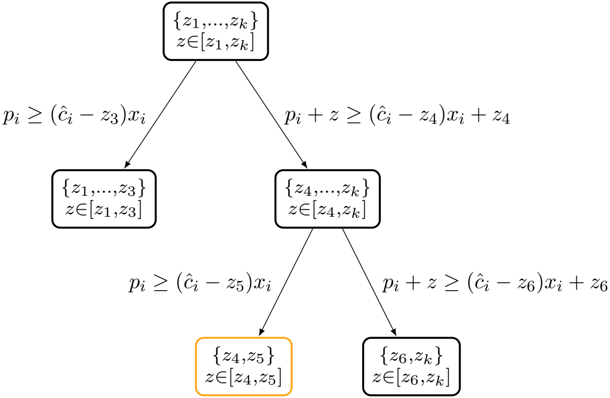

# Branch & Bound

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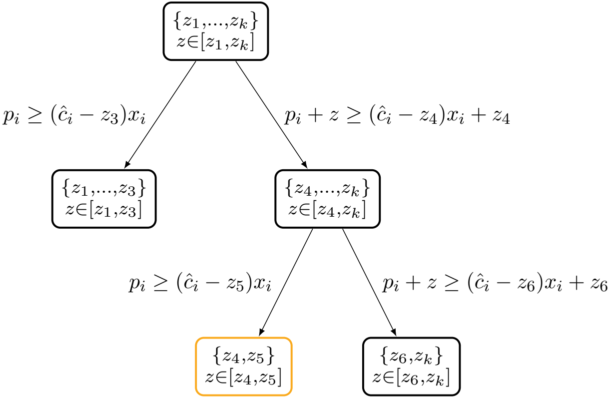
  - fast pruning of non-optimal $z$
  - bounds on $z$ yield many more structural properties



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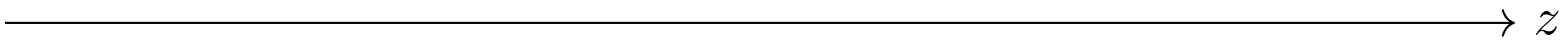

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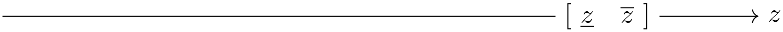

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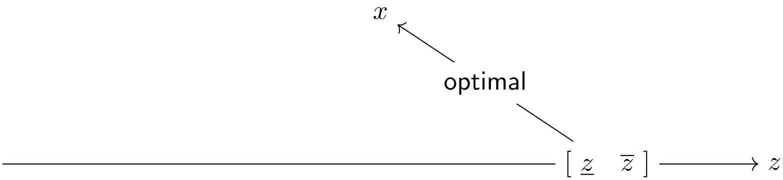

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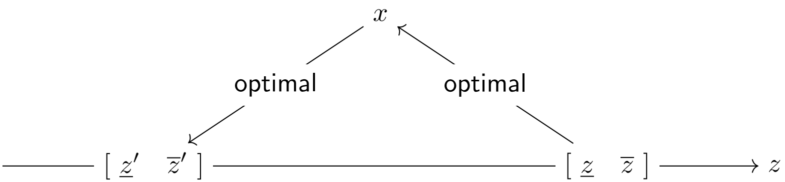

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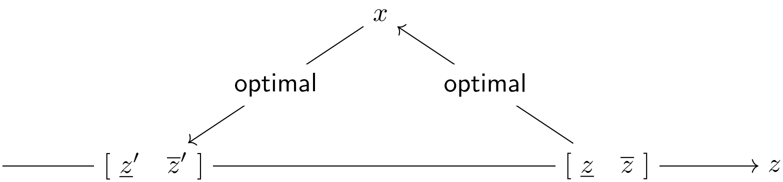


prohibiting $x$ for $z \in [\underline{z}, \overline{z}]$ leads to better dual bounds

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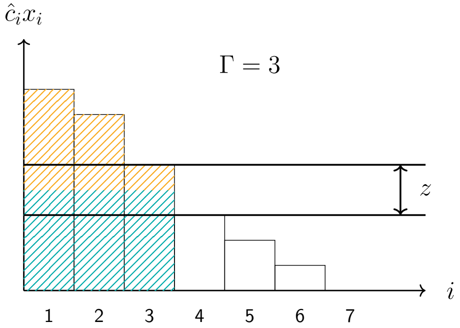

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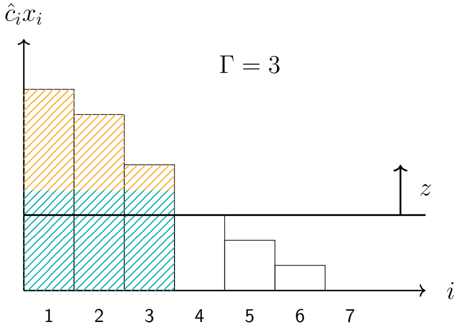

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$$\sum_{i \in N: \hat{c}_i > z} x_i \leq \Gamma$$

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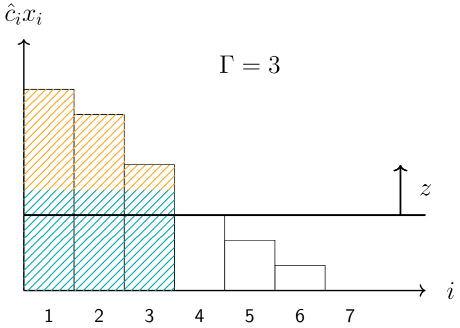

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$$\sum_{i \in N: \hat{c}_i > \overline{z}} x_i \leq \sum_{i \in N: \hat{c}_i > z} x_i \leq \Gamma$$

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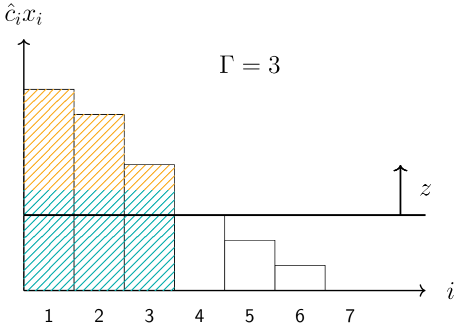

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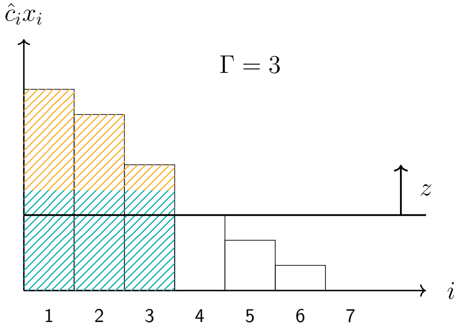

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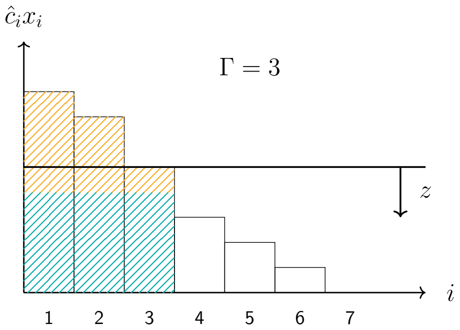

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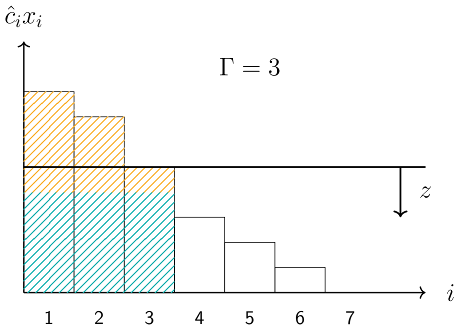

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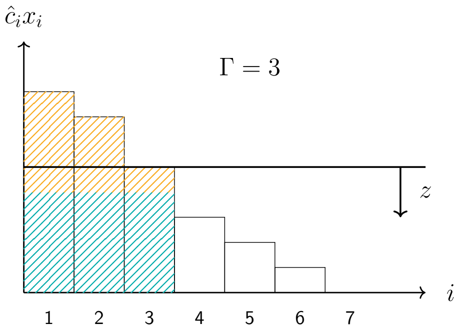

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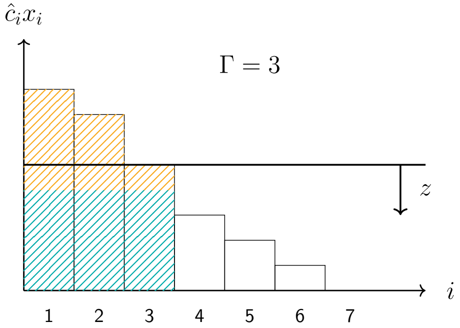

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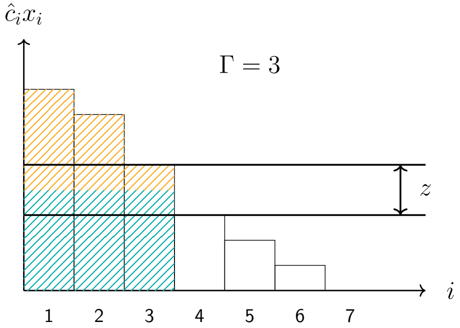


**Theorem**

*There is an optimal $z \in [\underline{z}, \overline{z}]$ for $x \in \{0,1\}^n$ iff the above inequalities are fulfilled.*

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# Computational Study

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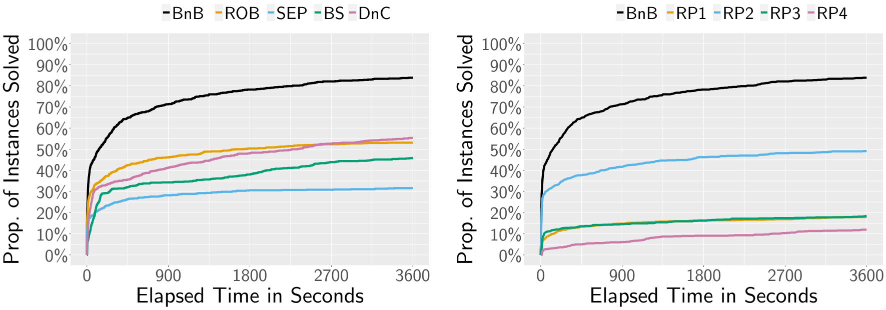
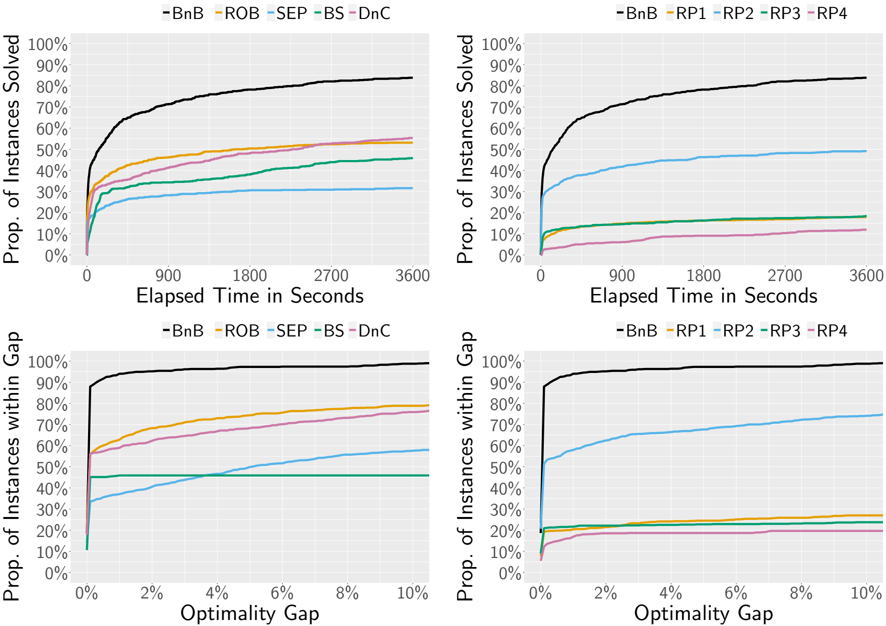
  - Bertsimas Sim standard reformulation, cutting-plane approach, Atamtürk's formulations, $|\mathcal{Z}|$ nominal subproblems, Divide & Conquer [HRS2018]

# Computational Results: B&B vs. Literature




BnB
ROB
SEP
BS
DnC
Prop. of Instances Solved
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
0
900
1800
2700
3600
Elapsed Time in Seconds
BnB
RP1
RP2
RP3
RP4
Prop. of Instances Solved
Elapsed Time in Seconds
BnB
ROB
SEP
BS
DnC
Prop. of Instances within Gap
0%
2%
4%
6%
8%
10%
Optimality Gap
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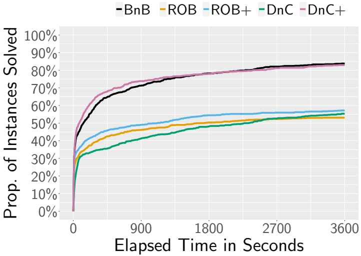


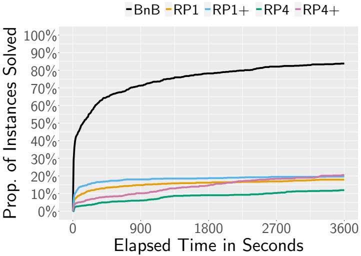

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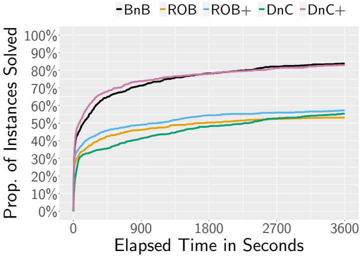


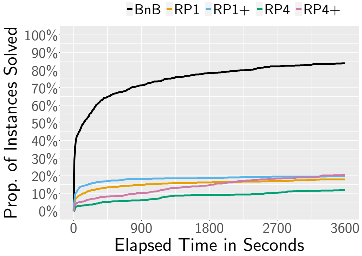


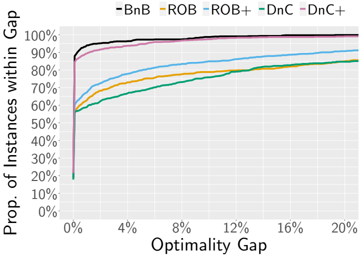


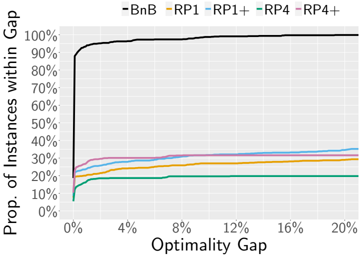

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# Recycling Valid Inequalities

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*Let $\sum_{i \in N} \pi_i x_i \leq \pi_0$ be a recyclable inequality. Then the* **recycled inequality**

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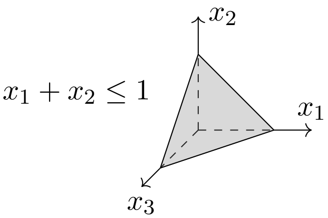

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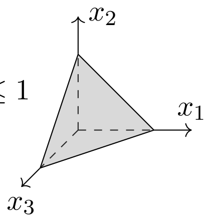

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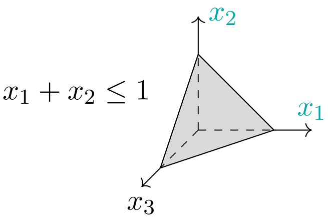

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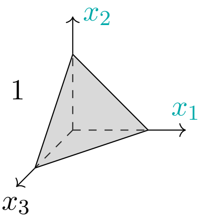

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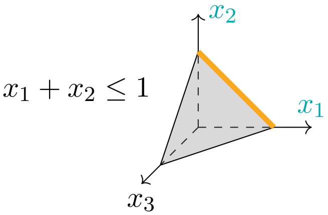

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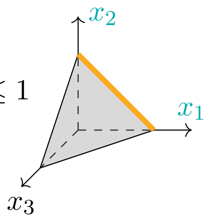

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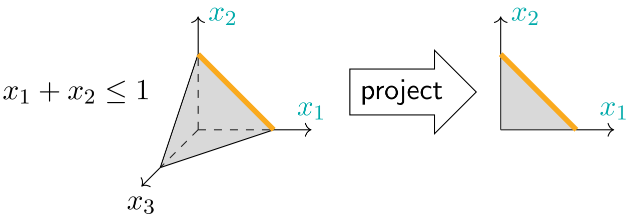

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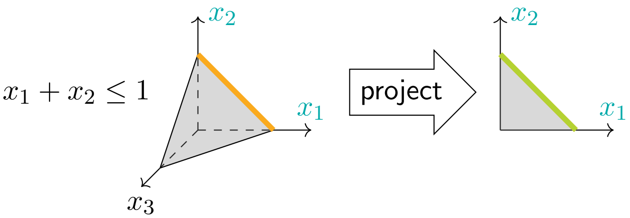

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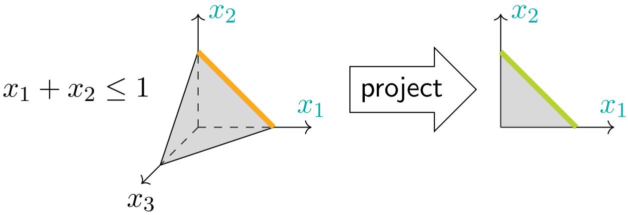


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# Facet-Defining Recycled Inequalities

**Corollary**

*Assume that* $\mathcal{C}^{\mathrm{NOM}}$ *is full-dimensional. If* $\sum_{i\in N} \pi_i x_i \leq \pi_0$ *is recyclable and facet-defining for* $\mathcal{C}^{\mathrm{NOM}}$*, then its recycled inequality is facet-defining for* $\mathcal{C}^{\mathrm{ROB}}$*.*

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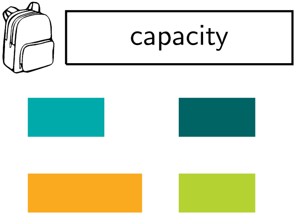
**Robust Knapsack**



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*Assume that* $\mathcal{C}^{\mathrm{NOM}}$ *is full-dimensional. If* $\sum_{i \in N} \pi_i x_i \leq \pi_0$ *is recyclable and facet-defining for* $\mathcal{C}^{\mathrm{NOM}}$*, then its recycled inequality is facet-defining for* $\mathcal{C}^{\mathrm{ROB}}$*.*

- corollary can be generalized to problems with lower dimension

**Observation**

Dominated inequalities can also yield facet-defining recycled inequalities.

**Robust Knapsack**

- let $\sum_{i \in C} x_i \leq |C| - 1$ be a minimal cover inequality

# Facet-Defining Recycled Inequalities

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Dominated inequalities can also yield facet-defining recycled inequalities.

## Robust Knapsack

- let $\sum_{i \in C} x_i \leq |C| - 1$ be a minimal cover inequality
- in general not facet-defining for knapsack
- but $\sum_{i \in C} p_i + (|C| - 1)z \geq \sum_{i \in C} \hat{c}_i x_i$ is always facet-defining for robust knapsack

- standard formulation + recycle constraints $\sum_{e \in \delta(v)} x_e \leq 1$ for all nodes $v$

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# Computational Study: Bipartite Matching

- standard formulation + recycle constraints $\sum_{e\in\delta(v)} x_e \leq 1$ for all nodes $v$
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| | robust standard formulation | | | | recycle constraints | | | |
|---|---|---|---|---|---|---|---|---|
| nodes | timeout | time | P-D integral | int. gap | timeout | time | P-D integral | int. gap |
| 50 | 0 | 1.73 | 0.04 | 19.53% | 0 | 0.48 | 0.04 | 0.33% |
| 100 | 9 | 2269.14 | 3.49 | 22.82% | 0 | 4.50 | 0.16 | 0.32% |
| 150 | 7 | 2223.68 | 2.56 | 23.66% | 0 | 150.40 | 0.59 | 0.27% |

- standard formulation + recycle constraints $\sum_{e\in\delta(v)} x_e \leq 1$ for all nodes $v$
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- 10 random instances each for different numbers of nodes $n \in \{50, 100, 150\}$
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- $\sim 504$-times faster for $100$ nodes and $\sim 15$-times faster for $150$ nodes

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- $\sim 504$-times faster for 100 nodes and $\sim 15$-times faster for 150 nodes
- $\sim 22$-times smaller primal-dual integral for 100 nodes and $\sim 4$-times smaller for 150 nodes

| | robust standard formulation | | | | recycle constraints | | | |
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- uses many structural properties
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# Conclusion

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**Thank you for your attention!**