# Very Short Introduction to Stochastic Programming

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# An optimization problem

A standard optimization problem

$$\begin{aligned} \min_{u_0} \quad & L(u_0) \\ s.t. \quad & g(u_0) \leq 0 \end{aligned}$$

# An optimization problem with uncertainty

Adding uncertainty $\xi$ in the mix

$$\min_{u_0} \quad L(u_0, \xi)$$
$$s.t. \quad g(u_0, \xi) \leq 0$$

Remarks:

- $\xi$ is unknown. Two main way of modelling it:
  - $\xi \in \Xi$ with a known uncertainty set $\Xi$, and a pessimistic approach. This is the robust optimization approach (RO).
  - $\xi$ is a random variable with known probability law. This is the Stochastic Programming approach (SP).
- Cost is not well defined.
  - RO : $\max_{\xi \in \Xi} L(u, \xi)$.
  - SP : $\mathbb{E}\big[L(u, \xi)\big]$.
- Constraints are not well defined.
  - RO : $g(u, \xi) \leq 0, \qquad \forall \xi \in \Xi$.
  - SP : $L(u, \xi) \leq 0, \qquad \mathbb{P} - a.s.$.

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# One-Stage Problem

Assume that $\boldsymbol{\Xi}$ as a discrete distribution, with $\mathbb{P}(\boldsymbol{\xi} = \xi_i) = p_i$ for $i \in [\![1, n]\!]$. Then, the one-stage problem

$$\begin{aligned} \min_{u_0} \quad & \mathbb{E}\Big[L(u_0, \boldsymbol{\xi})\Big] \\ s.t. \quad & g(u_0, \boldsymbol{\xi}) \leq 0, \qquad \mathbb{P} - a.s \end{aligned}$$

can be written

$$\begin{aligned} \min_{u_0} \quad & \sum_{i=1}^{n} p_i L(u_0, \xi_i) \\ s.t \quad & g(u_0, \xi_i) \leq 0, \qquad \forall i \in [\![1, n]\!]. \end{aligned}$$

# Recourse Variable

In most problem we can make a correction $u_1$ once the uncertainty is known:

$$u_0 \rightsquigarrow \xi_1 \rightsquigarrow u_1.$$

As the recourse control $u_1$ is a function of $\xi$ it is a random variable. The two-stage optimization problem then reads

$$\begin{aligned} \min_{u_0} \quad & \mathbb{E}\Big[L(u_0, \xi, \mathbf{u}_1)\Big] \\ s.t. \quad & g(u_0, \xi, \mathbf{u}_1) \leq 0, \qquad \mathbb{P} - a.s \end{aligned}$$

# Two-stage Problem

The extensive formulation of

$$\begin{aligned} \min_{u_0} \quad & \mathbb{E}\Big[L(u_0, \boldsymbol{\xi}, \mathbf{u}_1)\Big] \\ s.t. \quad & g(u_0, \boldsymbol{\xi}, \mathbf{u}_1) \leq 0, \qquad \mathbb{P}-a.s \end{aligned}$$

is

$$\begin{aligned} \min_{u_0, \{u_1^i\}_{i \in [\![1,n]\!]}} \quad & \sum_{i=1}^{n} p_i L(u_0, \xi_i, u_1^i) \\ s.t \quad & g(u_0, \xi_i, u_1^i) \leq 0, \qquad \forall i \in [\![1, n]\!]. \end{aligned}$$

# Multi-stage Problem

- We can consider a multi-stage problem with successive decision and aleas

$$u_0 \leadsto \xi_1 \leadsto u_1 \leadsto \xi_2 \leadsto \cdots u_T.$$

- If each each alea $\xi_i$ has 10 possible realizations, then there are
  - 1 control $u_0$
  - 10 control $u_1^i$
  - 100 control $u_2^i$
  - ...
- In practice only two or three-stage problem can be solved by Stochastic Programming approaches.
- Remark : a stage is not necessarily a time-step.