The geometry of balanced games

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- We show that it is a polyhedron, and find its vertices and extremal rays.

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- One of the best known solution: the *core* (Gillies, 1953)

$$C(v) = \{x \in \mathbb{R}^N : x(S) \geq v(S) \forall S, x(N) = v(N)\}$$

(coalitional rationality, or stability of the grand coalition N)

TU-games in other domains

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- In combinatorial optimization, when v is submodular, it can be seen as the rank function of a matroid. Then the (anti-)core of v is the *base polyhedron of v* (Edmonds, 1970).

Balanced collections

- (Shapley, 1967) A collection $\mathcal{B} \subseteq 2^N$ of nonempty coalitions is called *balanced* if there exist positive numbers λ_S for all $S \in \mathcal{B}$ s.t.
$$\sum_{S \in \mathcal{B}} \lambda_S 1^S = 1^N$$
(i.e., for every $i \in N$, $\sum_{S \ni i, S \in \mathcal{B}} \lambda_S = 1$)($1^N$ is in the relative interior of the cone generated by the 1^S, $S \in \mathcal{B}$).

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- Balanced collections correspond to *regular hypergraphs*

Nonemptiness of the core

Theorem (Bondareva-Shapley, sharp form)

A game v has a nonempty core if and only if for any minimal balanced collection $\mathcal{B}$ with balancing vector $(\lambda_S^{\mathcal{B}})_{S\in\mathcal{B}}$, we have

$$\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) \leq v(N).$$

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Games satisfying this condition are called *balanced*

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$\rightarrow$ We focus on $\mathcal{BG}_+(n)$ and $\mathcal{BG}(n)$.
Notation: $\mathfrak{B}^*(n)$: set of m.b.c. on N, except $\{N\}$.

Structure of $\mathcal{BG}_+(n)$

- $\mathcal{BG}_+(n)$ is determined by the following system of inequalities

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) \leqslant 1, \quad \mathcal{B} \in \mathfrak{B}^*(n)$$
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Theorem

Let $\mathcal{D}$ be a family of subsets $\mathcal{D}$ in $2^N \setminus \{\emptyset, N\}$. Then, $\mathcal{D}$ defines a vertex of $\mathcal{BG}_+(n)$ iff either $\mathcal{D} = \emptyset$ or $\bigcap \mathcal{D} \neq \varnothing$.

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n	1	2	3	4	5	6	7	8
v_n	1	3	19	471	162631	12884412819	$6.456e+19$	$1.361e+39$

Adjacency in $\mathcal{BG}_+(n)$

Recall that two vertices v_1, v_2 are *not adjacent* if there exist $\lambda, \lambda' \in [0, 1]$ and vertices v_3, v_4 distinct from v_1, v_2 s.t.

$$\lambda v_1 + (1 - \lambda) v_2 = \lambda' v_3 + (1 - \lambda') v_4$$

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Definition

(Naddef and Pulleyblank, 1981) A polytope $\mathcal{P}$ is said to be *combinatorial* if the two following conditions hold:

- All vertices of $\mathcal{P}$ are 0,1-valued.
- Given two vertices v_1, v_2 of $\mathcal{P}$, if they are not adjacent, then there exists two other different vertices v_3, v_4 such that

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As a consequence, the graph of the vertices of $\mathcal{BG}_+(n)$ is Hamiltonian ($n > 2$) or a hypercube ($n = 1, 2$).

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Consider two vertices v_1, v_2 *of* $\mathcal{BG}_+(n)$*, associated to* $\mathcal{D}_1, \mathcal{D}_2$ *respectively, and* $\bigcap \mathcal{D}_1 = \{i\} = \bigcap \mathcal{D}_2$*. Then* v_1 *and* v_2 *are adjacent iff either* $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *or the converse, and* $|\mathcal{D}_1 \Delta \mathcal{D}_2| = 1$*.*

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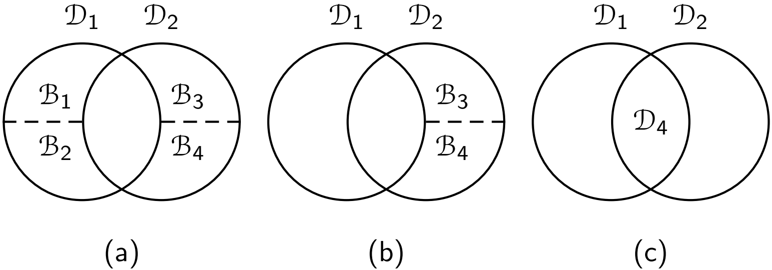

Figure: Non-adjacency of v_1, v_2, with associated collections $\mathcal{D}_1, \mathcal{D}_2$. Case (a): $\mathcal{D}_3 = \mathcal{B}_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2) \cup \mathcal{B}_3$, $\mathcal{D}_4 = \mathcal{B}_2 \cup (\mathcal{D}_1 \cap \mathcal{D}_2) \cup \mathcal{B}_4$; Case (b): $\mathcal{D}_3 = \mathcal{D}_1 \cup \mathcal{B}_3$, $\mathcal{D}_4 = (\mathcal{D}_1 \cap \mathcal{D}_2) \cup \mathcal{B}_4$ (similar when $\mathcal{D}_1, \mathcal{D}_2$ exchanged); Case (c): $\mathcal{D}_3 = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{D}_4 = \mathcal{D}_1 \cap \mathcal{D}_2$.

Structure of $\mathcal{BG}(n)$

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Let $n \geqslant 2$. Then $\mathcal{BG}(n)$ is $(2^n - 1)$-dimensional polyhedral cone, which is not pointed. Its lineality space $\mathrm{Lin}(\mathcal{BG}(n))$ *has dimension n, with basis* $(w_i)_{i \in N}$, $w_i = u_{\{i\}}$, *the unanimity game centered on* $\{i\}$

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As $\mathcal{BG}(n)$ is not pointed, it can be decomposed as follows:

$$\mathcal{BG}(n) = \mathrm{Lin}(\mathcal{BG}(n)) \oplus \mathcal{BG}^0(n)$$

where $\mathcal{BG}^0(n)$ is a supplementary space (not unique), chosen so that the coordinates corresponding to singletons are zero.

Structure of $\mathcal{BG}(n)$

Theorem

Let $n \geqslant 2$. The extremal rays of $\mathcal{BG}(n)$ are

- *The $2n$ extremal rays corresponding to* $\mathrm{Lin}(\mathcal{BG}(n))$*:* $w_1, \ldots, w_n, -w_1, \ldots, -w_n$*;*
- $2^n - n - 2$ *extremal rays of the form* $r_S = -\delta_S$, $S \subset N$, $|S| > 1$*;*
- n *extremal rays of the form*

$$r_i = \sum_{S \ni i, |S| > 1} \delta_S, \quad i \in N.$$

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Lemma

The cores of w_i, $-w_i$, r_i, r_S for all $i \in N$, $S \subset N$, $|S| > 1$ are singletons (respectively, $\{1^{\{i\}}\}$, $\{-1^{\{i\}}\}$, $\{1^{\{i\}}\}$, $\{0\}$).

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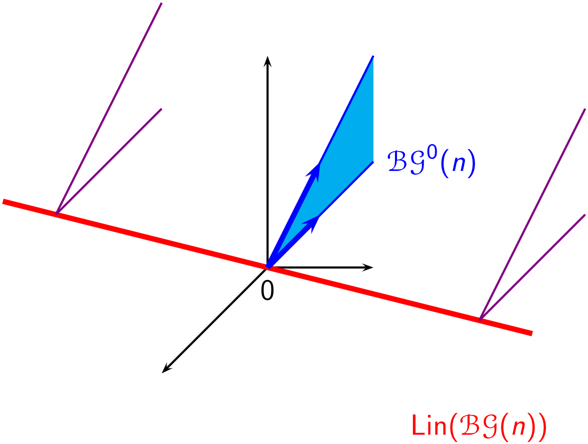

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What can we say more?

General result: a game in the interior of $\mathcal{BG}_+(n)$ (or $\mathcal{BG}(n)$) does not have a point core.

When is the core reduced to a point? Case of $\mathcal{BG}_+(n)$

Theorem

Consider two adjacent vertices v_1, v_2 of $\mathcal{BG}_+(n)$, with associated collections $\mathcal{D}_1, \mathcal{D}_2$ respectively, and $\bigcap \mathcal{D}_1 = \{i\}$, $\bigcap \mathcal{D}_2 = \{j\}$. Consider $v = \lambda v_1 + (1-\lambda) v_2$. Then:

1. *If $i = j$, then $C(v)$ is a singleton, i.e., v has a point core.*
2. *If $i \neq j$ and $n \leqslant 4$, then v has a point core.*

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When $n \geqslant 5$, taking two adjacent vertices v_1, v_2 having a point core does not guarantee that any game on the edge between v_1, v_2 has a point core. A more specific result seems difficult to obtain.

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Lemma

Any game in the lineality space $\mathcal{BG}(n)$ has a point core.

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We recall that facets of $\mathcal{BG}(n)$ are in bijection with the elements of $\mathfrak{B}^*(n)$, i.e., minimal balanced collections.

Theorem

Consider a m.b.c. $\mathcal{B} \in \mathfrak{B}^*(n)$ and its corresponding facet in $\mathcal{BG}(n)$.

1. *If $|\mathcal{B}| = n$, every game in the facet has a point core.*
2. *Otherwise, no game in the relative interior of the facet has a point core.*

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Theorem

Consider a face $\mathcal{F}$ of $\mathcal{BG}(n)$, being the intersection of facets $\mathcal{F}_1, \ldots, \mathcal{F}_p$ with associated m.b.c. $\mathcal{B}_1, \ldots, \mathcal{B}_p$. Then any game in $\mathcal{F}$ has a point core iff the rank of the matrix $\{1^S, S \in \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p\}$ is n.

The case $n = 3$

The lineality space has basis $\{u_{\{1\}}, u_{\{2\}}, u_{\{3\}}\}$, with extremal rays $-\delta_{12}, -\delta_{13}, -\delta_{23}$, and r_1, r_2, r_3.

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$\mathcal{B}_2 = \{1, 23\}$	$\times$	$\times$			$\times$	$\times$
$\mathcal{B}_3 = \{2, 13\}$	$\times$		$\times$	$\times$		$\times$
$\mathcal{B}_4 = \{3, 12\}$		$\times$	$\times$	$\times$	$\times$	
$\mathcal{B}_5 = \{12, 13, 23\}$				$\times$	$\times$	$\times$

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