Advances in Two-Stage Robust Optimization

An Augmented Lagrangian Duality Viewpoint

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Paris, 2024

Outline

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Robust Optimization

$$\min_{x \in X} \max_{\xi \in \Xi} \psi(x; \xi)$$

Here and now	Uncertainty	Wait and see
Make decision $x \in X$ based on *a priori* knowledge $\mathbb{P}(\boldsymbol{\xi} \in \Xi) > 0$	Observe the actual outcome $\bar{\xi}$ of $\boldsymbol{\xi}$	Endorse decision x no matter $\bar{\xi}$

time

Two-Stage Robust Optimization

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} \psi(x, y; \xi)$$

Here and now	Uncertainty	Wait and see
Make decision $x \in X$ based on *a priori* knowledge $\mathbb{P}(\xi \in \Xi) > 0$	Observe the actual outcome $\bar{\xi}$ of $\boldsymbol{\xi}$	Make recourse decision $y \in Y(x, \bar{\xi})$ based on *a posteriori* knowledge $\bar{\xi}$

$\longrightarrow$ time

Outline

Literature Review: Exact Approaches for Continuous Recourse

2004	Affine Decision Rules (approximation) Ben-Tal et al.	Duality
2009	Benders Decomposition Thiele et al.	Duality
2013	Column-and-Constraint Generation (CCG) Zeng et al.	Duality
2016	Benders & CCG without complete recourse Ayoub et al.	Duality
2023	Benders & CCG for convex problems Lefebvre et al.	Duality

Duality Usage: Turn Max into Min

Example: Static Robust Optimization

$$(a+\xi)^\top x \leq b \qquad \forall \xi \in \Xi$$

Duality Usage: Turn Max into Min

Example: Static Robust Optimization

$$(a+\xi)^\top x \leq b \qquad \forall \xi \in \Xi \quad \iff \quad a^\top x + \max_{\xi \in \Xi} \xi^\top x \leq b$$

Duality Usage: Turn Max into Min

Example: Static Robust Optimization

$$(a+\xi)^\top x \le b \qquad \forall \xi \in \Xi \qquad \Longleftrightarrow \qquad a^\top x + \max_{\xi \in \Xi} \xi^\top x \le b$$

$$\overset{\text{duality}}{\Longleftrightarrow} \qquad a^\top x + \min_{\lambda \in \Lambda(x)} f^\top \lambda \le b$$

Duality Usage: Turn Max into Min

Example: Static Robust Optimization

$$
\begin{aligned}
(a+\xi)^\top x \le b \qquad \forall \xi \in \Xi \quad &\Longleftrightarrow \quad a^\top x + \max_{\xi \in \Xi} \xi^\top x \le b \\
&\overset{\text{duality}}{\Longleftrightarrow} \quad a^\top x + \min_{\lambda \in \Lambda(x)} f^\top \lambda \le b \\
&\Longleftrightarrow \quad \exists \lambda \in \Lambda(x), \quad a^\top x + f^\top \lambda \le b
\end{aligned}
$$

Duality Usage: Move Things to the Objective Function

Example: Benders Decomposition

Say $Y(x, \xi) = \{y \in \mathbb{R}^n : Tx + Wy \geq h(\xi)\}$

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d^\top y$$

Duality Usage: Move Things to the Objective Function

Example: Benders Decomposition

Say $Y(x,\xi) = \{y \in \mathbb{R}^n : Tx + Wy \geq h(\xi)\}$

$$\min_{x\in X} \max_{\xi\in\Xi} \min_{y\in Y(x,\xi)} d^\top y \overset{\text{duality}}{=} \min_{x\in X} \max_{\xi\in\Xi} \max_{\lambda\in\Lambda} (h(\xi) - Tx)^\top \lambda$$

Duality Usage: Move Things to the Objective Function

Example: Benders Decomposition

Say $Y(x,\xi) = \{y \in \mathbb{R}^n : Tx + Wy \geq h(\xi)\}$

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\begin{aligned}
\min_{x\in X} \; \max_{\xi\in\Xi} \; \min_{y\in Y(x,\xi)} d^\top y \overset{\text{duality}}{=} & \min_{x\in X} \; \max_{\xi\in\Xi} \; \max_{\lambda\in\Lambda} (h(\xi) - Tx)^\top \lambda \\
= & \min \quad t \\
& \text{s.t.} \quad t \geq (h(\xi) - Tx)^\top \lambda \qquad \forall (\xi,\lambda) \in \Xi \times \Lambda \\
& \qquad\;\; x \in X
\end{aligned}
$$

Duality Usage: Move Things to the Objective Function

Example: Benders Decomposition

1. Solve the master problem

$$
\begin{aligned}
\min\ & t \\
\text{s.t. } & t \geq (h(\bar{\xi}^k) - Tx)^\top \lambda \qquad k = 1, \ldots, K \\
& x \in X
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Duality Usage: Move Things to the Objective Function

Example: Benders Decomposition

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$$\begin{aligned} &\min\ t \\ &\text{s.t. } t \geq (h(\bar{\xi}^k) - Tx)^\top \lambda \qquad k = 1, ..., K \\ &\qquad x \in X \end{aligned}$$

2. Solve the separation problem

$$\begin{aligned} &\max_{\lambda,\xi}\ (h(\xi) - T\bar{x})^\top \lambda \\ &\text{s.t. } \xi \in \Xi \\ &\qquad \lambda \in \Lambda \end{aligned}$$

Duality Usage: Move Things to the Objective Function

Example: Column-and-Constraint Generation

1. Solve the master problem

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Duality Usage: Move Things to the Objective Function

Example: Column-and-Constraint Generation

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Literature Review: Exact Approaches for Integer Recourse

2012*	Column-and-Constraint Generation (CCG) Zhao et al.	Min-Max-Min-Max Reformulation
2020	Branch-and-Cut Kämmerling et al.	Max-Min Inequality
2022	Branch-and-Price Arslan et al.	Polyhedral Result, Dantzig-Wolfe
2022	CCG Subramanyam	Min-Max-Min-Max Reformulation
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Objective Uncertainty

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Move Uncertainty to the Objective

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Problem Setting

We consider general MI(N)LPs

$$\begin{aligned} z^* = \min_x \ & c^\top x \\ \text{s.t. } & Ax = b \\ & Bx \geq f \\ & x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \end{aligned}$$

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Our main interest is in "dualizing" constraints "$Ax = b$"

Let $X = \{x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} : Bx \geq f\}$

Lagrangian Duality

Primal Problem

$$
\begin{aligned}
z^* = \min_{x} \ & c^\top x \\
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& x \in X
\end{aligned}
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Lagrangian Dual Problem

$$z^{\mathrm{LD}} = \sup_{\lambda \in \mathbb{R}^m} \ z^{\mathrm{LR}}(\lambda)$$

$$z^{\mathrm{LR}}(\lambda) = \min_{x \in X} \ c^\top x + \lambda^\top (Ax - b)$$

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Strong duality does not hold in general, i.e., $z^* > z^{\mathrm{LD}}$

Augmented Lagrangian Duality

Primal Problem

$$\begin{aligned} z^* = \min_x \ & c^\top x \\ \text{s.t. } & Ax = b \\ & x \in X \end{aligned}$$

Augmented Lagrangian Dual Problem

$$z_\rho^{\mathrm{LD}+} = \sup_{\lambda \in \mathbb{R}^m} \ z_\rho^{\mathrm{LR}+}(\lambda)$$

$$z_\rho^{\mathrm{LR}+}(\lambda) = \min_{x \in X} \ c^\top x + \lambda^\top (Ax - b) + \rho\psi(Ax - b)$$

with $\psi(u) > u$ if and only if $u \neq 0$, $\psi(0) = 0$

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Our main interest:

$z^* \stackrel{?}{=} z_\rho^{\mathrm{LD}+}$.

1. For $\rho \to \infty$? (Asymptotic result)

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Primal Problem

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Our main interest:

$$z^* \overset{?}{=} z_\rho^{\text{LD}+}.$$

1. For $\rho \to \infty$? (Asymptotic result)
2. For some $\rho < \infty$? (Exactness result)

Previous Works

Growing interest in the discrete community

	Asymptotic	Exactness	Poly. size	Poly. time	Opt. set
ILP (Boland and Eberhard 2014)	✓	✓	$\Uparrow$		$\Uparrow$
MILP (Feizollahi et al. 2016)	✓	✓	$\Uparrow$		✓
MIQP (Gu et al. 2020)	✓	✓	✓		
MICP (Bhardwaj et al. 2024)	✓	✓			

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MILP (Feizollahi et al. 2016)	✓	✓	$\Uparrow$	✓	✓
MIQP (Gu et al. 2020)	✓	✓	✓	✓*	$\Uparrow$
MICP (Bhardwaj et al. 2024)	✓	✓			$\Uparrow$
MINLP (our work)	✓	✓			✓

Assumptions

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X is compact.

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Assumption (Penalty Function)

The penalty function $\psi : \mathbb{R}^m \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is

1. *continuous on* $\mathrm{dom}(\psi)$, *i.e.,* $\lim_{u \to u^*} \psi(u) = \psi(u^*)$;
2. *positive definite, i.e.,* $\psi(u) > 0$ *for all* $u \neq 0$ *and* $\psi(0) = 0$.

Bounding the Augmenting Term

Let x_ρ denote any solution of

$$z_\rho^{\text{LR+}}(\lambda) = \min_{x \in X} \; c^\top x + \lambda^\top (Ax - b) + \rho\psi(Ax - b)$$

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Weak Duality

$$z_\rho^{\mathsf{LR}+}(\lambda) \leq z^*$$

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$$\begin{aligned} & c^\top x_\rho + \lambda^\top (Ax_\rho - b) + \rho\psi(Ax_\rho - b) \le c^\top x^* \\ \iff \quad & \rho\psi(Ax_\rho - b) \le c^\top x^* - c^\top x_\rho + \lambda^\top (b - Ax_\rho) \end{aligned}$$

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$$\boxed{\rho\psi(Ax_\rho - b) \leq \kappa(\mathcal{E}, A, b, c, \lambda)}$$

Asymptotic Result (Part 1)

Let $\varepsilon > 0$. Any $\rho \geq \frac{1}{\varepsilon}\kappa(\mathcal{E}, A, b, c, \bar{\lambda})$ guarantees that $\psi(Ax_\rho - b) \leq \varepsilon$.

$$
\begin{aligned}
& \rho \geq \frac{1}{\varepsilon}\kappa(\mathcal{E}, A, b, c, \bar{\lambda}) \\
\implies \quad & \rho\psi(Ax_\rho - b) \geq \frac{1}{\varepsilon}\kappa(\mathcal{E}, A, b, c, \bar{\lambda})\psi(Ax_\rho - b) \\
\implies \quad & \kappa(\mathcal{E}, A, b, c, \bar{\lambda}) \geq \frac{1}{\varepsilon}\kappa(\mathcal{E}, A, b, c, \bar{\lambda})\psi(Ax_\rho - b) \\
\implies \quad & \varepsilon \geq \psi(Ax_\rho - b)
\end{aligned}
$$

Asymptotic Result (Part 2)

Let $\varepsilon \to 0$ ($\rho \to \infty$)

There is a limit point to (a sub-sequence of) $(x_\rho)_{\rho>0}$, say $x^*_\infty \in X$

By continuity of ψ ...

$$\begin{aligned} & \varepsilon \geq \psi(Ax_\rho - b) \\ \overset{\varepsilon \to 0}{\Longrightarrow} \quad & 0 \geq \lim_{\rho \to \infty} \psi(Ax_\rho - b) = \psi(Ax^*_\infty - b) \\ \Longleftrightarrow \quad & Ax^*_\infty = b \end{aligned}$$

This shows that x^*_∞ is feasible for the primal problem!

$$z^* = \lim_{\rho \to \infty} z_\rho^{\mathrm{LR+}}(\lambda)$$

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$$z^* = \lim_{\rho\to\infty} z_\rho^{\text{LR+}}(\lambda) = \sup_{\rho>0} z_\rho^{\text{LR+}}(\lambda)$$

Norm Penalty Functions

We now assume $\psi = \|\cdot\|$ for any norm $\|\cdot\|$.

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Say that $\rho < \infty$ is exact for $\|\cdot\|$

By equivalence of norms, there exists $\gamma > 0$ such that $\|\cdot\| \leq \gamma\|\cdot\|'$

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Say that $\rho < \infty$ is exact for $\|\cdot\|$

By equivalence of norms, there exists $\gamma > 0$ such that $\|\cdot\| \leq \gamma \|\cdot\|'$

$$\begin{aligned} z^* &= \min_{x \in X} \; c^\top x + \lambda^\top (Ax - b) + \rho \|Ax - b\| \\ &\leq \min_{x \in X} \; c^\top x + \lambda^\top (Ax - b) + \rho\gamma \|Ax - b\|' \\ &\leq z^* \end{aligned}$$

Thus $\rho\gamma$ is exact for $\|\cdot\|'$

Norm Penalty Functions

We now assume $\psi = \|\cdot\|$ for any norm $\|\cdot\|$.

Lemma 1 The choice of norm does not matter

Lemma 2 The choice of λ does not matter

Norm Penalty Functions

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Lemma 1 The choice of norm does not matter

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Say $\rho < \infty$ is exact for $\|\cdot\|_2$ and $\lambda = 0$

Norm Penalty Functions

We now assume $\psi = \|\cdot\|$ for any norm $\|\cdot\|$.

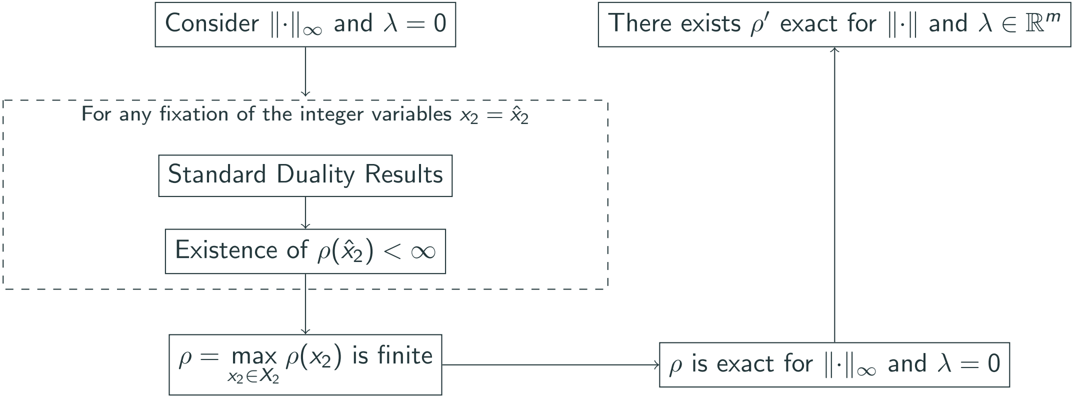
Lemma 1 The choice of norm does not matter

Lemma 2 The choice of λ does not matter

Say $\rho < \infty$ is exact for $\|\cdot\|_2$ and $\lambda = 0$

$$\begin{aligned}
z^* &= \min_{x \in X} \; c^\top x + \rho \|Ax - b\|_2 \\
&= \min_{x \in X} \; c^\top x + \bar{\lambda}^\top (Ax - b) - \bar{\lambda}^\top (Ax - b) + \rho \|Ax - b\|_2 \\
&\leq \min_{x \in X} \; c^\top x + \bar{\lambda}^\top (Ax - b) + \|\bar{\lambda}\|_2 \|Ax - b\|_2 + \rho \|Ax - b\|_2 \\
&= \min_{x \in X} \; c^\top x + \bar{\lambda}^\top (Ax - b) + (\|\bar{\lambda}\|_2 + \rho) \|Ax - b\|_2
\end{aligned}$$

Exact Penalty Parameter: The Overall Picture



Exact Penalty Parameter: The Proof

Fix the integer part in the primal, say, $x_2 = \hat{x}_2$ with $x = (x_1, x_2)$

$$
\begin{aligned}
z^*(\hat{x}_2) = \min_{x_1} \ & c_1^\top x_1 + c_2^\top \hat{x}_2 \\
\text{s.t. } & Ax_1 = b - A\hat{x}_2 \\
& g(x_1, \hat{x}_2) \leq 0 \\
& x_1 \in \mathbb{R}^{n_1}
\end{aligned}
$$

We distinguish two cases

1. Feasible case: $z^*(x_2) < \infty$
2. Infeasible case: $z^*(x_2) = \infty$

Feasible Case

As per the previous lemma, let's use $\psi(\cdot) = \|\cdot\|_\infty$ and $\lambda = 0$

$$
\begin{aligned}
z^*(\hat{x}_2) - c_2^\top \hat{x}_2 = \quad \min_{x_1} \quad & c_1^\top x_1 \\
\text{s.t.} \quad & A_1 x_1 = b - A_2 \hat{x}_2 \\
& B_1 x_1 \geq f - B_2 \hat{x}_2
\end{aligned}
$$

Feasible Case

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$$z^*(\hat{x}_2) - c_2^\top \hat{x}_2 = \begin{array}{rl} \min\limits_{x_1} & c_1^\top x_1 \\ \text{s.t.} & A_1 x_1 = b - A_2\hat{x}_2 \\ & B_1 x_1 \geq f - B_2\hat{x}_2 \end{array} = \begin{array}{rl} \max\limits_{\mu,\lambda} & (b - A_2\hat{x}_2)^\top \mu + (f - B_2 x_2)^\top \lambda \\ \text{s.t.} & A_1^\top \mu + B_1^\top \lambda = c_1 \\ & \lambda \geq 0 \end{array}$$

Feasible Case

As per the previous lemma, let's use $\psi(\cdot) = \|\cdot\|_\infty$ and $\lambda = 0$

$$z^*(\hat{x}_2) - c_2^\top \hat{x}_2 = \begin{array}{rl} \min\limits_{x_1} & c_1^\top x_1 \\ \text{s.t.} & A_1 x_1 = b - A_2 \hat{x}_2 \\ & B_1 x_1 \geq f - B_2 \hat{x}_2 \end{array} = \begin{array}{rl} \max\limits_{\mu,\lambda} & (b - A_2 \hat{x}_2)^\top \mu + (f - B_2 x_2)^\top \lambda \\ \text{s.t.} & A_1^\top \mu + B_1^\top \lambda = c_1 \\ & \lambda \geq 0 \end{array}$$

$$z_\rho^{\mathrm{LR+}}(\hat{x}_2) - c_2^\top \hat{x}_2 = \begin{array}{rl} \min\limits_{x_1} & c_1^\top x_1 + \rho \|A_1 x_1 + A_2 \hat{x}_2 - b\|_\infty \\ \text{s.t.} & B_1 x_1 \geq f - B_2 \hat{x}_2 \end{array}$$

Feasible Case

As per the previous lemma, let's use $\psi(\cdot) = \|\cdot\|_\infty$ and $\lambda = 0$

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z^*(\hat{x}_2) - c_2^\top \hat{x}_2 = \begin{array}{rl} \min\limits_{x_1} & c_1^\top x_1 \\ \text{s.t.} & A_1 x_1 = b - A_2 \hat{x}_2 \\ & B_1 x_1 \geq f - B_2 \hat{x}_2 \end{array} = \begin{array}{rl} \max\limits_{\mu,\lambda} & (b - A_2 \hat{x}_2)^\top \mu + (f - B_2 x_2)^\top \lambda \\ \text{s.t.} & A_1^\top \mu + B_1^\top \lambda = c_1 \\ & \lambda \geq 0 \end{array}
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$$

Any $\rho > \|\mu^*\|_1$ is large enough!

Infeasible Case

$$\text{“}z_\rho^{\mathrm{LR}+} = z^* = \min_{x_2} z_\rho^{\mathrm{LR}+}(\hat{x}_2)\text{”}$$

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Any ρ such that $z_\rho^{\mathrm{LR}+}(\hat{x}_2) > \mathrm{UB}$ is large enough!

Infeasible Case

$$\text{“}z_\rho^{\mathrm{LR+}} = z^* = \min_{x_2} z_\rho^{\mathrm{LR+}}(\hat{x}_2)\text{”}$$

Any ρ such that $z_\rho^{\mathrm{LR+}}(\hat{x}_2) > \mathrm{UB}$ is large enough!

$$\begin{aligned}
\min_{\rho,\mu,\lambda} \ & \rho \\
\text{s.t. } & (b - A_2\hat{x}_2)^\top \mu + (f - B_2 x_2)^\top \lambda \geq \mathrm{UB} \\
& A_1^\top \mu + B_1^\top \lambda = c_1 \\
& \lambda \geq 0 \\
& \|\mu\|_1 \leq \rho
\end{aligned}$$

Outline

Outline

Arslan and Detienne (2021)

Assumption: $Y(x, \xi) = Y(x) = \{y \in Y : Tx + Wy \leq h\}$ with $Y \subseteq \mathbb{R}^{n-q} \times \mathbb{Z}^q_{\geq 0}$

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x)} d(\xi)^\top y$$

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Main Contributions:

1. Decomposition-based relaxation
2. Algorithmic scheme to solve the relaxation
3. Special case in which the relaxation is tight

1. Decomposition-Based Relaxation

$$
\begin{aligned}
\min_{x} \ \max_{\xi \in \Xi} \ \min_{y} \quad & d(\xi)^\top y \\
\text{s.t.} \quad & y \in Y \\
& Tx + Wy \leq h
\end{aligned}
$$

1. Decomposition-Based Relaxation

$$\begin{aligned} \min_{x}\ \max_{\xi\in\Xi}\ \min_{y}\quad & d(\xi)^\top y \\ \text{s.t.}\quad & y \in Y \\ & Tx + Wy \le h \end{aligned} \quad = \quad \begin{aligned} \min_{x}\ \max_{\xi\in\Xi}\ \min_{y}\quad & d(\xi)^\top y + \rho\,[Tx + Wy - h]^+ \\ \text{s.t.}\quad & y \in Y \end{aligned}$$

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$$\ge \min_{x} \ \max_{\xi \in \Xi} \ \min_{y \in \mathrm{conv}(Y)} \quad d(\xi)^\top y + \rho \left[Tx + Wy - h \right]^+$$

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$$

$$
\begin{aligned}
= \min_{x,y} \quad & \min_{\lambda \in \Lambda(y)} g^\top \lambda \\
\text{s.t.} \quad & x \in X \\
& y \in \mathrm{conv}(Y) \\
& Tx + Wy \le h
\end{aligned}
$$

2. Algorithmic Scheme to Solve the Relaxation

$$\text{conv}(Y) = \left\{ \sum_{e=1}^{E} \alpha_e \bar{y}^e : \sum_{e=1}^{E} \alpha_e = 1, \quad \alpha_e \geq 0 \right\}$$

1. Solve a restricted master problem

$$\begin{aligned} \min_{x,y,\lambda} \ & g^\top \lambda \\ \text{s.t. } & x \in X \\ & y = \sum_{e=1}^{E'} \alpha_e \bar{y}^e && (\lambda \in \mathbb{R}^n) \\ & \sum_{e=1}^{E'} \alpha_e = 1 && (\pi \in \mathbb{R}) \\ & \alpha_e \geq 0, \qquad e = 1, \ldots, E' \\ & Tx + Wy \leq h \\ & \lambda \in \Lambda(y) \end{aligned}$$

2. Solve a pricing problem

$$\begin{aligned} \min_{y} \ & \pi + \mu^\top y \\ \text{s.t. } & y \in Y \end{aligned}$$

3. Special Case in Which the Relaxation is Tight

Additional Assumption: $Y(x, \xi) = Y(x) = \{y \in Y : x + y \leq 1\}$ and $X \subseteq \{0,1\}^n$

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For a fixed x decision, the penalty term is linear!

$$[x + y - 1]^+ = \left\{ \begin{array}{ll} y_i, & \text{if } x_i = 1 \\ [y_i - 1]^+ = 0, & \text{if } x_i = 0 \end{array} \right\} = x^\top y$$

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$$\begin{aligned} \min_{x} \max_{\xi \in \Xi} \min_{y} \quad & d(\xi)^\top y \\ \text{s.t.} \quad & y \in Y \\ & x + y \le 1 \end{aligned} \quad = \min_{x} \max_{\xi \in \Xi} \min_{y \in Y} d(\xi)^\top y + \rho\, [x + y - 1]^+$$

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Further Works on Objective Uncertainty

- Kämmerling et al. (2021), Alternative approach based on locally valid cutting planes
- Bodur et al. (2024), Handle constraint "$y \in \text{conv}(Y)$" using decision diagrams
- Detienne et al. (2024), Introduce spatial branching to deal with general MINLPs

Outline

Subramanyam (2022)

Assumptions:

- $\Xi \subseteq \{0,1\}^p$
- $Y(x,\xi) = \{y \in Y : Tx + Wy \le h(\xi)\}$

$$\min_{x\in X} \max_{\xi\in\Xi} \min_{y\in Y(x,\xi)} d(\xi)^\top y$$

Main Contributions:

1. Equivalence between constraint and objective uncertainty
2. Min-max-min-max reformulation
3. A nested column-and-constraint generation algorithm

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d(\xi)^\top y$$

1. Equivalence Between Constraint and Objective Uncertainty

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d(\xi)^\top y = \min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y} d(\xi)^\top y + \rho[Tx + Wy - h(\xi)]^+$$

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$$\begin{aligned} = \min_{x \in X} \max_{\xi \in \Xi} \min_{y,z} \quad & d(\xi)^\top y \\ \text{s.t.} \quad & Tx + Wy \geq h(z) \\ & y \in Y \\ & 0 \leq z \leq 1 \\ & z = \xi \end{aligned}$$

$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d(\xi)^\top y = \cancel{\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y} d(\xi)^\top y + \rho[Tx + Wy - h(\xi)]^+}$$

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For a fixed scenario ξ, the penalty term is linear!

$$\|z - \xi\|_1 = \left\{\begin{matrix} z_i & \text{if } \xi_i = 0 \\ 1 - z_i & \text{if } \xi_i = 1 \end{matrix}\right\} = e^\top \xi + (e - 2\xi)^\top z$$

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$$\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d(\xi)^\top y = \min_{x \in X} \max_{\xi \in \Xi} \begin{array}[t]{rl} \min\limits_{y,z} & d(\xi)^\top y + \rho(e^\top \xi + (e - 2\xi)^\top z) \\ \text{s.t.} & Tx + Wy \geq h(z) \\ & y \in Y \\ & 0 \leq z \leq 1 \end{array}$$

1. Equivalence Between Constraint and Objective Uncertainty

For a fixed scenario ξ, the penalty term is linear!

$$\|z-\xi\|_1 = \begin{Bmatrix} z_i & \text{if } \xi_i = 0 \\ 1 - z_i & \text{if } \xi_i = 1 \end{Bmatrix} = e^\top \xi + (e - 2\xi)^\top z$$

$$\min_{x\in X} \max_{\xi\in\Xi} \min_{y\in Y(x,\xi)} d(\xi)^\top y = \min_{x\in X} \max_{\xi\in\Xi} \min_{(y,z)\in Z(x)} d(\xi)^\top y + \rho(e^\top \xi + (e-2\xi)^\top z)$$

2. Min-Max-Min-Max Reformulation

$$
\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d(\xi)^\top y = \begin{array}[t]{rl} \displaystyle\min_{x \in X} \max_{\xi \in \Xi} \min_{y,z} & d(\xi)^\top y + \rho(e^\top \xi + (e - 2\xi)^\top z) \\ \text{s.t.} & Tx + Wy \geq h(z) \\ & y \in Y \\ & 0 \leq z \leq 1 \end{array}
$$

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$$
\begin{aligned}
\min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y(x,\xi)} d(\xi)^\top y = \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y,z} \quad d(\xi)^\top y + \rho(e^\top\xi + (e-2\xi)^\top z) \\
& \qquad\qquad\qquad \text{s.t.} \quad Tx + Wy \geq h(z) \\
& \qquad\qquad\qquad\qquad\quad y \in Y \\
& \qquad\qquad\qquad\qquad\quad 0 \leq z \leq 1 \\
= \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y}\ \min_{z} \quad d(\xi)^\top y + \rho(e^\top\xi + (e-2\xi)^\top z) \\
& \qquad\qquad\qquad\qquad\quad \text{s.t.} \quad Tx + Wy \geq h + Hz \\
& \qquad\qquad\qquad\qquad\qquad\quad 0 \leq z \leq 1
\end{aligned}
$$

2. Min-Max-Min-Max Reformulation

$$
\begin{aligned}
\min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y(x,\xi)}\ d(\xi)^\top y = \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y,z} && d(\xi)^\top y + \rho(e^\top \xi + (e-2\xi)^\top z) \\
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& && \qquad\ \ 0 \leq z \leq 1 \\
= \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y}\ \max_{\lambda,\mu} && d(\xi)^\top y + \rho e^\top \xi + (h - Wy - Tx)^\top \lambda + e^\top \mu \\
& && \text{s.t.}\quad -H^\top \lambda + \mu \leq \rho(e - 2\xi) \\
& && \qquad\ \ \lambda, \mu \leq 0
\end{aligned}
$$

2. Min-Max-Min-Max Reformulation

$$
\begin{aligned}
\min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y(x,\xi)}\ d(\xi)^\top y = \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y,z} && d(\xi)^\top y + \rho(e^\top\xi + (e-2\xi)^\top z) \\
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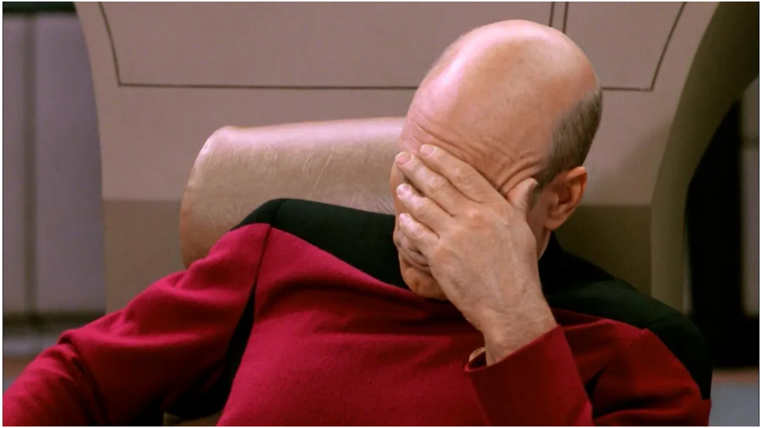
= \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y}\ \min_{z} && d(\xi)^\top y + \rho(e^\top\xi + (e-2\xi)^\top z) \\
& && \text{s.t.} \quad Tx + Wy \geq h + Hz \\
& && \qquad 0 \leq z \leq 1 \\
= \ & \min_{x\in X}\ \max_{\xi\in\Xi}\ \min_{y\in Y}\ \max_{(\lambda,\mu)\in\Lambda(\xi)} && f(x,\xi,y,\lambda,\mu)
\end{aligned}
$$

2. Min-Max-Min-Max Reformulation



3. A Nested Column-and-Constraint Generation Algorithm

Column-and-Constraint Generation Algorithm for $\min_{x \in X} \max_{\xi \in \Xi} \min_{y \in Y(x,\xi)} d(\xi)^\top y$

1. Solve the master problem

$$\begin{aligned} \min\ & t \\ \text{s.t. } & t \geq d(\bar{\xi}^k)^\top y^k && k = 1, \ldots, K \\ & y^k \in Y(x, \bar{\xi}^k) && k = 1, \ldots, K \\ & x \in X \end{aligned}$$

2. Solve the separation problem

$$\max_{\xi \in \Xi} \min_{y \in Y(\bar{x},\xi)} d(\xi)^\top y$$

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& y^k \in Y(x, \bar{\xi}^k) && k = 1, ..., K \\
& x \in X
\end{aligned}
$$

2. Solve the separation problem

$$
\max_{\xi \in \Xi} \min_{y \in Y(\bar{x},\xi)} d(\xi)^\top y = \max_{\xi \in \Xi} \min_{y \in Y} \max_{(\lambda,\mu) \in \Lambda(\xi)} f(x, \xi, y, \lambda, \mu)
$$

Nested CCG Algorithm for $\max_{\xi \in \Xi} \min_{y \in Y} \max_{(\lambda,\mu) \in \Lambda(\xi)} f(\bar{x}, \xi, y, \lambda, \mu)$

1. Solve the master problem

$$\begin{aligned} &\min\ s \\ &\text{s.t. } s \geq f(\bar{x}, \xi, \bar{y}^j, \lambda^j, \mu^j) \qquad j = 1, \dots, J \\ &\qquad (\lambda^j, \mu^j) \in \Lambda(\xi) \qquad j = 1, \dots, J \\ &\qquad \xi \in \Xi \end{aligned}$$

3. A Nested Column-and-Constraint Generation Algorithm

Nested CCG Algorithm for $\max\limits_{\xi \in \Xi} \min\limits_{y \in Y} \max\limits_{(\lambda,\mu) \in \Lambda(\xi)} f(\bar{x}, \xi, y, \lambda, \mu)$

1. Solve the master problem

$$\begin{aligned} &\min\ s \\ &\text{s.t. } s \geq f(\bar{x}, \xi, \bar{y}^j, \lambda^j, \mu^j) \qquad j = 1, \ldots, J \\ &\qquad (\lambda^j, \mu^j) \in \Lambda(\xi) \qquad j = 1, \ldots, J \\ &\qquad \xi \in \Xi \end{aligned}$$

2. Solve the separation problem

$$\min_{y \in Y} \max_{(\lambda,\mu) \in \Lambda(\xi)} f(\bar{x}, \bar{\xi}, y, \lambda, \mu)$$

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2. Solve the separation problem

$$\min_{y \in Y} \max_{(\lambda,\mu) \in \Lambda(\xi)} f(\bar{x}, \bar{\xi}, y, \lambda, \mu) = \min_{y \in Y(\bar{x}, \bar{\xi})} d(\bar{\xi})^\top y$$

Further Work on Constraint Uncertainty and CCG

1. Tsang et al. (2023). Inexact CCG algorithm.
2. Lefebvre et al. (2024). Alternative approach based on locally valid cutting planes.
3. Lefebvre et al. (2024). Combining Branch-and-Price and CCG.

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Augmented Lagrangian Duality

Outline

Conclusion

- ALD can be used to move things to the objective function in integer problems

  Just like standard duality for continuous problems
- ALD leads to simplified proofs for recent approaches in the literature
- There are still open questions!
  - How to compute small penalty parameters?
  - Approaches for continuous uncertainty sets? Can ALD help?
  - Can we use ALD in bilevel optimization as well?

Preprint: Exact Augmented Lagrangian Duality for Nonconvex Mixed-Integer Nonlinear Optimization