Envy-free division of a cake with groups, and other extensions

Frédéric Meunier

École des Pontsa

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Joint work with Ayumi Igarashi

Dividing a cake

Theorem (Folklore)

To divide a cake between two people in an envy-free manner, let one person cut the cake and let the other choose.

Plan

Standard setting

Envy-free cake division

A cake has to be shared between people.

It will be divided into as many pieces as there are people.

Each person will be assigned a piece.

Envy-free division: each person prefers his piece.

Envy-free cake division

A cake has to be shared between people.

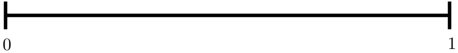
It will be divided into as many pieces as there are people.

Each person will be assigned a piece.

Envy-free division: each person is at least as happy with his piece than with any other piece.

Model

- n players: $j = 1, \ldots, n$

- Cake: $[0,1] =$ 

- Division of the cake: partition $\mathcal{I}$ of $[0,1]$ into nonempty intervals (the pieces)

- Player j has a choice function:

$$c_j : \{\text{divisions}\} \to 2^{\{\text{pieces}\}} .$$

Given a division $\mathcal{I}$, player j is happy with the pieces $I \in \mathcal{I}$ such that $I \in c_j(\mathcal{I})$.

- Given a division $\mathcal{I}$, an envy-free assignment is $\pi : \{\text{players}\} \longrightarrow \{\text{pieces}\}$
  - $\star$ π is bijective.
  - $\star$ $\pi(j) \in c_j(\mathcal{I})$ for every player j.

Existence of envy-free divisions

- Choice function c_j is closed if

$$\lim_{k\to\infty} \mathcal{I}^k = \mathcal{I} \text{ and } I^k \in c_j(\mathcal{I}^k)\ \forall k \quad \Longrightarrow \quad I^\infty \in c_j(\mathcal{I})$$

- Choice function c_j is hungry if

$$I \in c_j(\mathcal{I}) \quad \Longrightarrow \quad \lambda(I) \neq 0$$

Theorem Stromquist, Woodall 1980

No matter how many players there are, when all choice functions are closed and hungry, there is always an envy-free division.

Algorithmic consideration

Theorem Deng–Qi–Saberi 2012

For every fixed number $k \geqslant 3$ of players with hungry choice functions, computing an (approximate) envy-free division is PPAD-complete.

Theorem Deng–Qi–Saberi 2012

Suppose there are 3 players and the choice function are hungry and monotone. Then computing an (approximate) envy-free division can be done in $O(\log^2 1/\varepsilon)$.

Monotonicity: Consider a division $\mathcal{I}$, a player j, and a piece $I \in \mathcal{I}$ such that $I \in c_j(\mathcal{I})$. For any new division $\mathcal{I}'$ with $I' \supseteq I$ and $K' \subseteq K$ for all other pieces $K \neq I$, we have $I' \in c_j(\mathcal{I}')$.

Plan

Group extension

Cake division among groups

Theorem Segal-Halevi–Suksompong 2021

Consider an instance with n players. Let $k_1, \ldots, k_q$ be nonnegative integers summing up to n. When all choice functions are closed and hungry, there exist a division into q pieces and a partition of the players into q groups of size $k_1, \ldots, k_q$ with an envy-free assignment of the pieces to the q groups.

Motivation

- Public basketball court
- 30 players want to play on some day
- Cake = the day
- Players have different preferences regarding the time at which they prefer to play

With the theorem:

It is possible to partition the players into 3 groups of 10 players each, and divide the day into 3 contiguous intervals—one interval per group—so that each group of 10 is happy to play in its designated time slot.

Algorithms

Counterpart to groups of the polynomial result of Deng, Qi, and Saberi (2012).

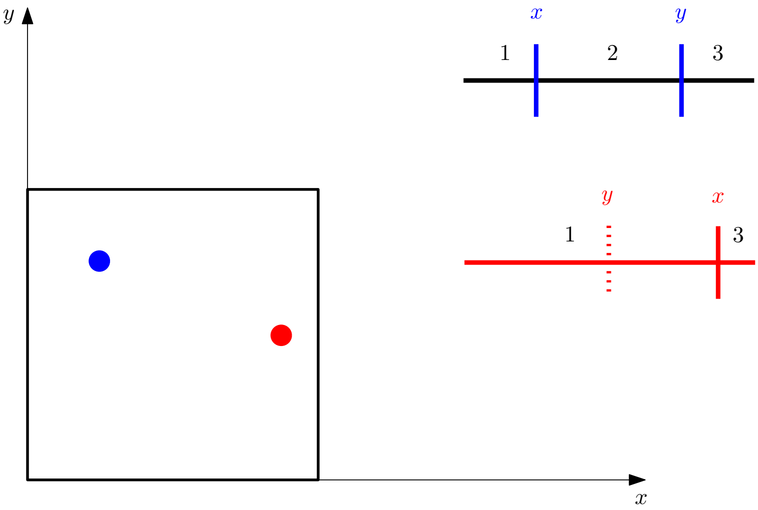
Theorem Igarashi–M. 2023+

Suppose there are n players and the choice function are hungry and monotone. Then computing 3 groups and an (approximate) envy-free division can be done in $O(n \log^2 1/\varepsilon)$.

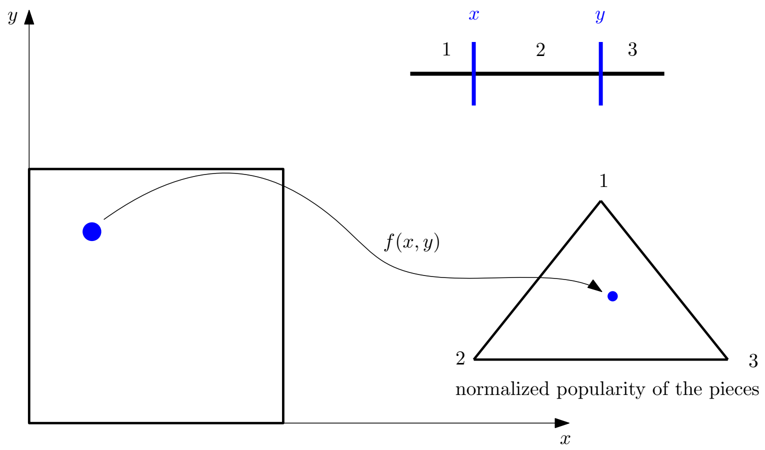
Plan

Two-dimensional topology

Configuration space



Measuring the popularity



In maths

For cuts located at x and y

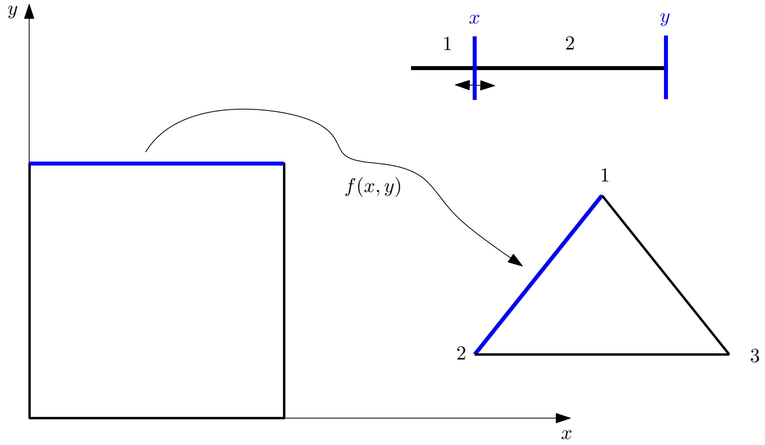
$$f_i(x, y) := \frac{1}{n} \times (\#\text{players prefering } i)$$

$$f_1(x, y) + f_2(x, y) + f_3(x, y) = 1\,.$$

With:

- $f = (f_1, f_2, f_3)$
- $\square = \{(x, y) \colon x, y \in [0, 1]\}$
- $\triangle = \{(z_1, z_2, z_3) \in \mathbb{R}_+ \colon z_1 + z_2 + z_3 = 1\}$

$$f : \square \to \triangle$$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3
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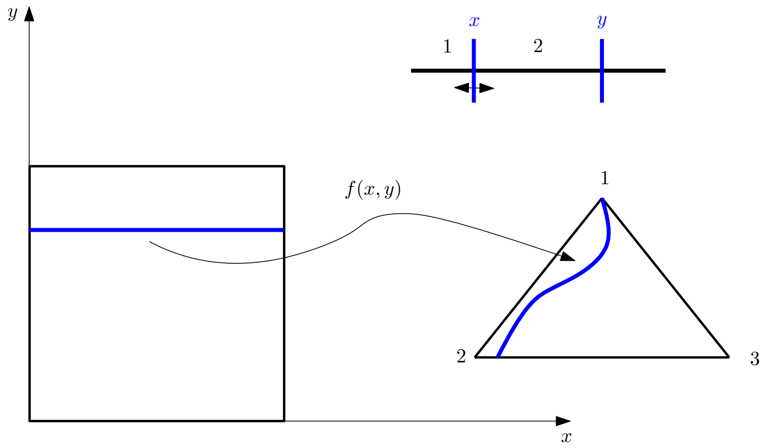

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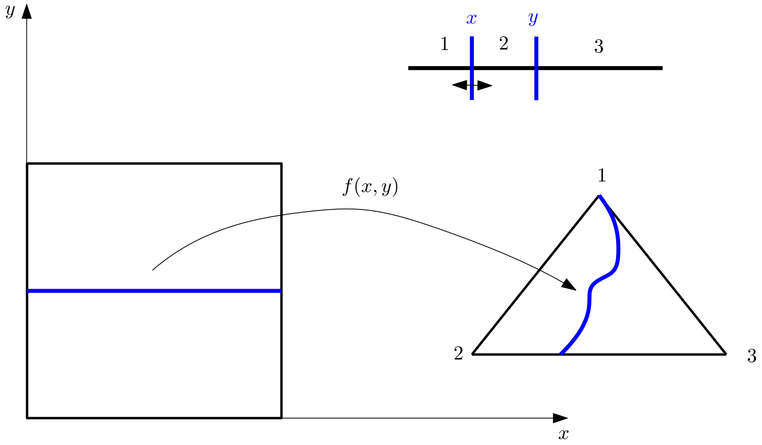

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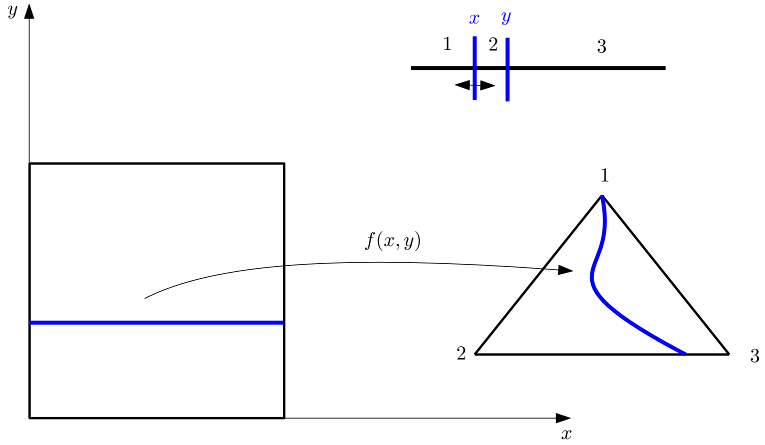

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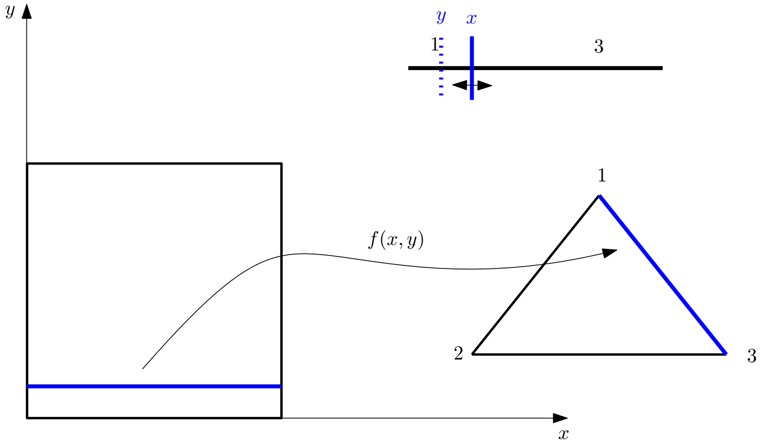
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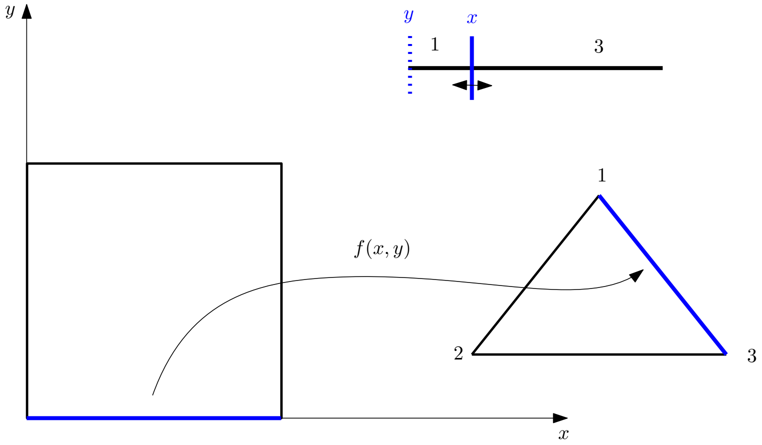
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f(x, y)
2
3
y
x

Finishing the proof

Lemma

The map f is surjective.

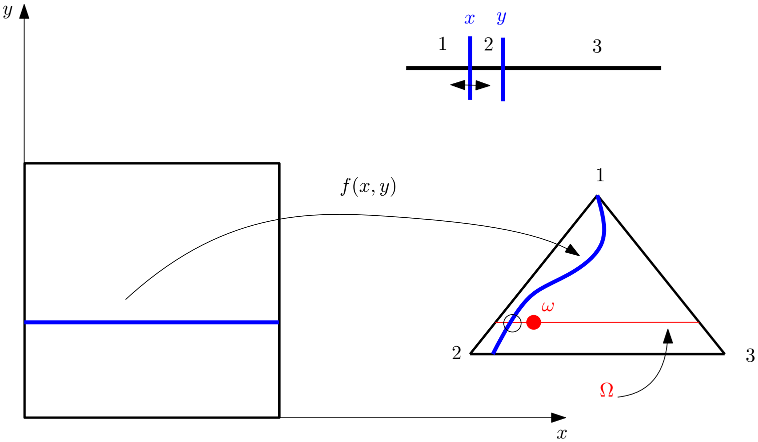
Let $\omega = (k_1/n, k_2/n, k_3/n)$.

In particular, there exists $(x^*, y^*) \in \square$ such that $f(x^*, y^*) = \omega$.

This means:
$(x^*, y^*) =$ division into 3 pieces for which it exists a partition of the players into 3 groups of size k_1, k_2, k_3 with an envy-free assignment.

Plan

Algorithms


y
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f(x, y)
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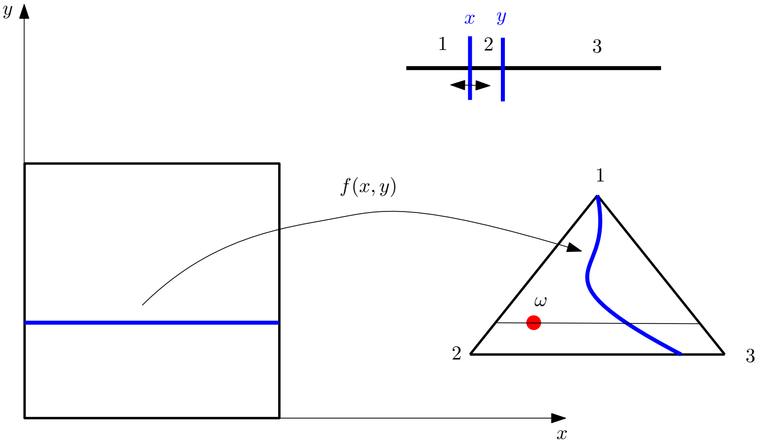


y
x
y
1
2
3
f(x, y)
1
ω
2
3
x

Computing the intersection

Lemma "Horizontal-monotonicity"

If $x \leqslant x'$, then $f_1(x, y) \leqslant f_1(x', y)$.

Up to a polynomially computable perturbation, intersection well-defined

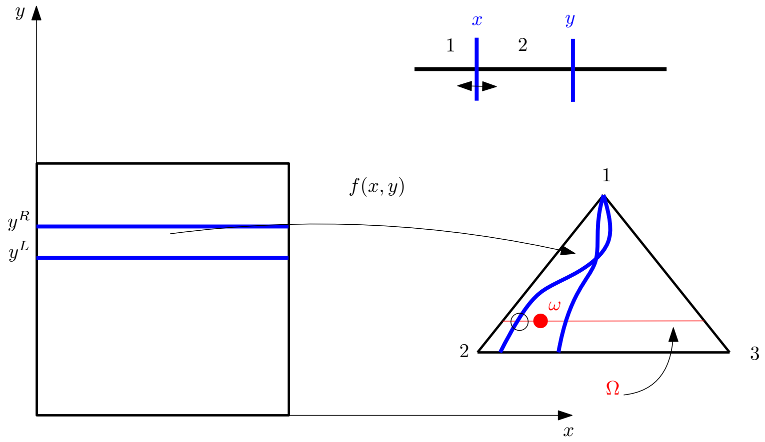
$\Longrightarrow$ binary search computing $f\left(\{(x, y)\colon x \in [0, 1]\}\right) \cap \Omega$ for any fixed $y \in [0, 1]$ in $O(n \log 1/\varepsilon)$

Strip containing ω

$\Longrightarrow$ second binary search computing y^L and y^R such that

- $|y^R - y^L| = \varepsilon$
- $f\big(\{(x, y^L) \colon x \in [0,1]\}\big) \cap \Omega$ is on the left of ω
- $f\big(\{(x, y^R) \colon x \in [0,1]\}\big) \cap \Omega$ is on the right of ω

Complexity $= O(n \log^2 \varepsilon)$



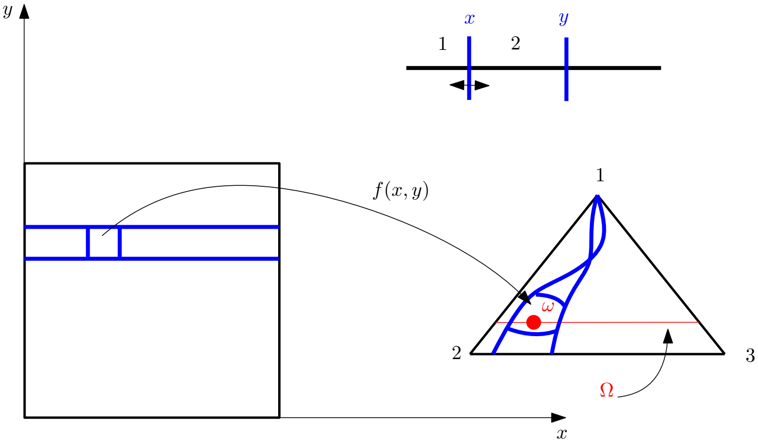
y
x
y
1
2
y^R
y^L
$f(x,y)$
1
ω
2
3
Ω
x

Locating ω

Last binary search: identify a small $(1/\varepsilon \times 1/\varepsilon)$-square "whose image by f" contains ω.

This is an "approximate" envy-free division.

Complexity still $O(n \log^2 \varepsilon)$



y
x
y
1
2
f(x, y)
1
ω
2
3
Ω
x

Affine extension and approximate division

♣ Actually, it is not really the image by f.

♣ We have a small square with vertices v_1, v_2, v_3, v_4 such that $\omega \in \mathrm{conv}\big(f(v_1), f(v_2), f(v_3), f(v_4)\big)$.

♣ In other words, there exist nonnegative $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\sum_\ell \alpha_\ell = 1$ s.t.

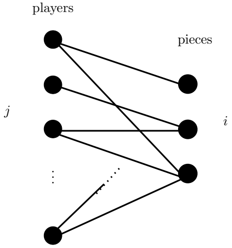
$$\sum_\ell \alpha_\ell f_i(v_\ell) = \frac{k_i}{n}\,.$$

Finishing the proof

Let $w_{ji} := \sum_\ell \alpha_\ell \mathbf{1}(\text{player } j \text{ prefers piece } i \text{ at } v_\ell)$. Then:

$$w_{j1} + w_{j2} + w_{j3} = 1 \ \forall j \quad \text{and} \quad \sum_{j=1}^{n} w_{ji} = k_i \ \forall i\,.$$

Bipartite graph H



Edges ji correspond to $w_{ji} > 0$.
We want $F \subseteq E(H)$ such that

- $\deg_F(j) = 1$ for all j
- $\deg_F(i) = k_i$ for all i

total unimodularity of

$$\left\{ \boldsymbol{x} \in \mathbb{R}_+^E : \sum_{e \in \delta_H(j)} x_e = 1 \, \forall j \in [n], \quad \sum_{e \in \delta_H(i)} x_e = k_i \, \forall i \in \{1,2,3\} \right\} \quad \text{QED}$$

Plan

Further extensions

Birthday and poison

Birthday player:
classical extension of cake-cutting,
does not share his preferences

Non-hungry player:
recent extension, might prefer a
piece of length 0

Birthday and poison

Theorem Woodall 1980

Consider an instance with n players, one of them being a "birthday" player. There exists a division into n pieces such that, no matter which piece is chosen by the "birthday" player, there is an envy-free assignment of the remaining pieces to the $n-1$ players.

Theorem Avvakumov–Karasev 2020

Consider an instance with n players, with closed choice functions. If n is a prime power, then there exists an envy-free division.

Birthday, bad cake, groups

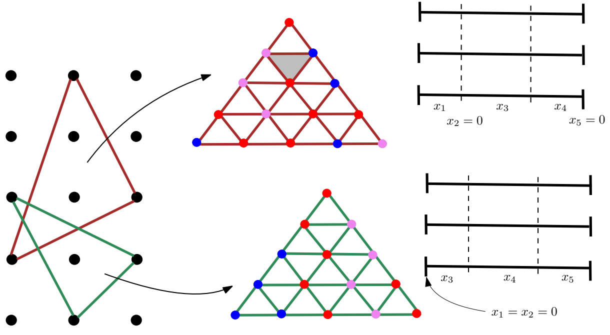
Theorem Igarashi–M. 2023

Consider an instance of a cake with n players, one of them being a "birthday" player, with closed choice functions. Let q be an integer such that $q \leqslant n$. If q is a prime power, then there exists a division into q pieces so that no matter which piece is chosen by the "birthday" player, there is an envy-free assignment of the remaining pieces with each piece assigned to the same number of players (up to one player).

Here, an envy-free assignment is $\pi \colon \{\text{players}\} \longrightarrow \{\text{pieces}\}$

- ⋆ $|\pi^{-1}(\text{piece } I)| \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$ for every piece I.
- ⋆ $\pi(j) \in c_j(\mathcal{I})$ for every player j.

Main tool



Chessboard complex $\triangle_{2n-1,n}$

Theorem Volovikov 1980

Let p be a prime number and $G = \big((\mathbb{Z}_p)^k, +\big)$. Consider a G-invariant triangulation of $\Delta_{2n-1,n}$, whose vertices are G-equivariently labeled with elements of G. Then there is a fully labeled simplex.

Combinatorial optimization

Lemma

Let $a_1, a_2, \ldots, a_q$ be nonnegative real numbers summing up to $n-1$, and $H = ([n-1], [q]; E)$ a bipartite graph with nonnegative edge weights w_e. If

$$\sum_{e \in \delta_H(j)} w_e = 1 \quad \forall j \in [n-1] \quad \text{and} \quad \sum_{e \in \delta_H(i)} w_e = a_i \quad \forall i \in [q]\,,$$

then for every i^*, there is an assignment $\pi\colon [n-1] \to [q]$ s.t.

- for each $j \in [n-1]$, the vertex $\pi(i)$ is a neighbor of i in H,
- for each $i \in [q] \setminus \{i^*\}$, we have $|\pi^{-1}(i)| \in \{\lfloor a_i \rfloor, \lceil a_i \rceil\}$,
- $|\pi^{-1}(i^*)| = \lfloor a_{i^*} \rfloor$.

Proof. polytope

$$\left\{ \boldsymbol{x} \geqslant \mathbf{0} \colon \sum_{e \in \delta_H(j)} x_e = 1 \, \forall j \in [n-1] \text{ and } \lfloor a_i \rfloor \leqslant \sum_{e \in \delta_H(i)} x_e \leqslant \lceil a_i \rceil \, \forall i \in [q] \right\},$$

total unimodularity, carefully chosen extreme point of the polytope

Thank you