On the existence of population monotonic allocation schemes for families of operations research games

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Operations Research Games

(P. Borm, H. Hamers, and R. Hendrickx. Operations research games: A survey. Top, 9 (2001): 139-199.)

- Cooperative games based on a (discrete) structure that underlies a combinatorial optimisation problem.
- Players control parts of the underlying system (e.g., vertices, edges, resource bundles, jobs)
- In working together the players can possibly create extra gains or save costs.
- how to share the extra revenues or cost savings?

Population Monotonic Allocation Schemes

(Y. Sprumont (1990) Population monotonic allocation schemes for cooperative games with transferable utility. Games and Economic behavior 2.4: 378-394.).

- Objective of a PMAS: providing a condition of dynamic stability to guarantee that once a coalition S has decided upon an allocation of $u(S)$, no player wish to form a coalition included in S
- our goal: prove whether a PMAS exists (or not) for many ORGs.

Outline

1. Cooperative games in short
2. Generalized Additive Games and PMAS
3. ORGs as GAGs
   - Weighted glove games
   - Link connection games
   - Weighted minimum coloring games
4. Furture directions (games on matroids)

Basics

A Transferable Utility (TU) game is a tuple (N, v) where

- $N = \{1, 2, ..., n\}$ is the set of players
- $v : 2^N \to \mathbb{R}$ is its characteristic function

By convention, $v(\emptyset) = 0$.
A game (N, v) is called

- monotonic if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$;
- superadditive if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.

The subgame corresponding to some coalition $T \subseteq N$ is the game

$$(T, v_T)$$

with $v_T(S) = v(S)$ for all $S \subseteq T$.

Basics

The core of a TU game (N, v) is the set

$$C(v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \quad \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}$$

A game (N, v) is called

- balanced if it has a nonempty core;
- totally balanced if the core of every subgame is nonempty;
- convex if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T$ and $i \in N \setminus T$, or, equivalently,
  supermodular if
  $$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$
  for all $S, T \in 2^N$.

PMAS

Given a TU game (N, v), the table

$$x = (x_i^S)_{\emptyset \neq S \in 2^N, i \in S}$$

is said to be a Population Monotonic Allocation Scheme (PMAS) (Sprumont (1990)) if

(i) *efficiency:* For all $S \subseteq N$, $S \neq \emptyset$, $\sum_{i \in S} x_i^S = v(S)$.

(ii) *monotonicity:* For all $S \subseteq T$ and for all $i \in S$, $x_i^S \leq x_i^T$.

Observe that each row (x_i^S) of a PMAS is in the core of the subgame v_S for all S.

So, a game with a PMAS is also a totally balanced game.

Convex games and PMAS

Convex games have PMAS (Sprumont (1990); see also Ichiishi (1981), Shapley (1971)).

Example

Consider the game $(\{1,2,3\}, v)$ such that $v(1) = v(3) = 0, v(2) = 3, v(1,2) = 3, v(1,3) = 1, v(2,3) = 4, v(1,2,3) = 5$.

S	$\phi_1^\sigma(v)$	$\phi_2^\sigma(v)$	$\phi_3^\sigma(v)$
$\{1,2,3\}$	0	3	2
$\{1,2\}$	0	3	*
$\{1,3\}$	0	*	1
$\{2,3\}$	*	3	1
$\{1\}$	0	*	*
$\{2\}$	*	3	*
$\{3\}$	*	*	0

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Convex games have PMAS (Sprumont (1990); direct consequences of previous results Ichiishi (1981), Shapley (1971)).

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S	$\phi_1^\sigma(v)$	$\phi_2^\sigma(v)$	$\phi_3^\sigma(v)$
$\{1,2,3\}$	$\frac{1}{2}$	$\frac{7}{2}$	1
$\{1,2\}$	0	3	*
$\{1,3\}$	$\frac{1}{2}$	*	$\frac{1}{2}$
$\{2,3\}$	*	$\frac{7}{2}$	$\frac{1}{2}$
$\{1\}$	0	*	*
$\{2\}$	*	3	*
$\{3\}$	*	*	0

The Shapley value of convex games is PMAS extendible.

A totally balanced (ToBa) game without PMAS

Consider $(\{1,2,3,4\}, v)$ such that $v(1,2,3,4) = 2$, $v(S) = 1$ if $|S| = 3$, $v(1,3) = v(1,4) = v(2,3) = v(2,4) = 1$ and $v(S) = 0$ otherwise (check it is ToBa; Sprumont (1990)). Suppose the following scheme:
Suppose a PMAS exists:

S	1	2	3	4
$\{1,2,3,4\}$	x_1	x_2	x_3	x_4
$\{1,2,3\}$	0	0	1	*
$\{1,2,4\}$	0	0	*	1
$\{1,3,4\}$	1	*	0	0
$\{2,3,4\}$	*	1	0	0
...	...	...	...	

$$x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, x_4 \geq 1 \;\Rightarrow\; x_1 + x_2 + x_3 + x_4 \geq 4 > v(1,2,3,4)$$

Impossible!

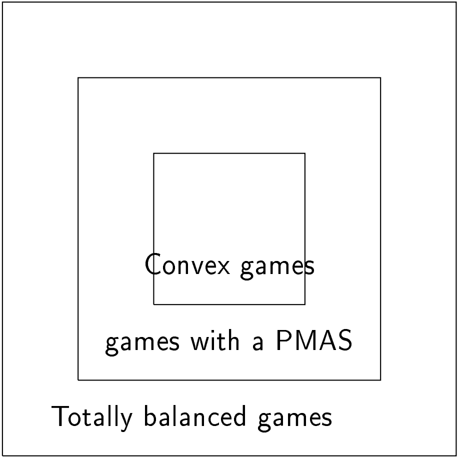
A game with PMAS that is not convex

Consider $(\{1,2,3\}, v)$ such that $v(1,2,3) = v(1,3) = v(2,3) = 1$ and $v(S) = 0$ otherwise This game is not convex: $v(1,2,3) - v(2,3) = 1 - 1$ and $v(1,3) - v(3) = 1 - 0$.

The unique PMAS is:

S	$\phi_1^\sigma(v)$	$\phi_2^\sigma(v)$	$\phi_3^\sigma(v)$
$\{1,2,3\}$	0	0	1
$\{1,2\}$	0	0	*
$\{1,3\}$	0	*	1
$\{2,3\}$	*	0	1
$\{1\}$	0	*	*
$\{2\}$	*	0	*
$\{3\}$	*	*	0

TU-games



Generalized Additive Games (Cesari et al. IJGT(2017))

Definition

We shall call *Generalized Additive Situation* (GAS) any triple $\langle N, w, \mathcal{M} \rangle$, where:

- N is a set of players;
- $w \in \mathbb{R}^N_+$ a vector of positive real numbers;
- $\mathcal{M} : 2^N \to 2^N$, is a *coalitional map, which assigns a coalition* $\mathcal{M}(S)$ *to each coalition* $S \subseteq N$ *of players (with* $\mathcal{M}(\emptyset) = \emptyset$*).*

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Definition

Given the GAS $\langle N, w, \mathcal{M} \rangle$, the associated *Generalized Additive Game* (GAG) is the TU-game $(N, v_{\mathcal{M},w})$ such that $v_{\mathcal{M},w}(\emptyset) = 0$ and for $S \neq \emptyset$:

$$v_{\mathcal{M},w}(S) = \sum_{i \in \mathcal{M}(S)} w_i$$

Many ORGs from the literature are GAGs

Examples

- airport games (Littlechild and Owen (1973); Littlechild and Thompson (1977)); generalized airport games (Norde et al. (2002))
- maintenance games (Koster (1999))
- peer games (Branzei et al. 2002)
- link-connection games (Nagamochi et al. (1997), Moretti (2017))
- minimum coloring games (Deng et al. (2000), Hamers et al. (2014))
- games on mountain situations (Moretti et al. 2002)
- argumentation games (Bonzon et al. 2014)
- connectivity games (Amer and Giménez 2004; Lindelauf et al. 2013
- "centrality" games (Michalak et al. (2013))
- Simple mcst games (Norde et al. (2004))
- many other TU-games (simple games, weighted glove games, etc...)

Properties for coalitional maps

A *coalitional map* $\mathcal{M} : 2^N \to 2^N$ such that $\mathcal{M}(\emptyset) = \emptyset$ is called:

1) *monotonic* if $\mathcal{M}(S) \subseteq \mathcal{M}(T)$ for every $S, T \in 2^N$ with $S \subseteq T$;
2) *proper* if $\mathcal{M}(S) \cap \mathcal{M}(T) = \emptyset$ for every $S, T \in 2^N$ with $S \cap T = \emptyset$;
3) *veto-rich* if for every $i \in N$ we either have $i \notin \mathcal{M}(S)$ for every $S \in 2^N$ or $i \in \mathcal{M}(N)$ and $\cap\{S : i \in \mathcal{M}(S)\} \neq \emptyset$;
4) *supermodular* if $\mathcal{M}(S) \cap \mathcal{M}(T) = \mathcal{M}(S \cap T)$ for every $S, T \in 2^N$.

Characterization for PMAS

Moretti, S., Norde, H. (2021) Some new results on generalized additive games. Int J Game Theory. https://doi.org/10.1007/s00182-021-00786-w

Theorem

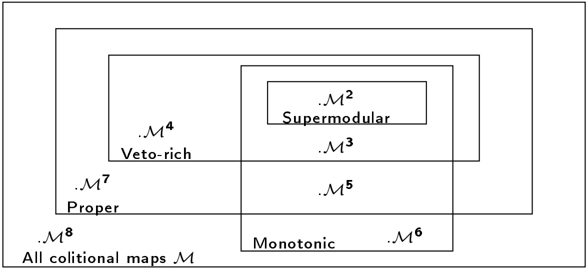
The following statements are equivalent:

I) $\mathcal{M}$ *is veto-rich and monotonic;*

II) $(N, v_{\mathcal{M},w})$ *admits a pmas for every* $w \in \mathbb{R}^N_+$*;*

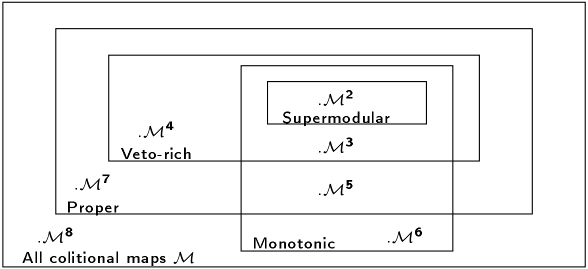
III) $(N, v_{\mathcal{M},w})$ *is totally balanced for every* $w \in \mathbb{R}^N_+$*.*

Euler diagram



(a) sets of coalitonal maps $\mathcal{M}$

Euler diagram



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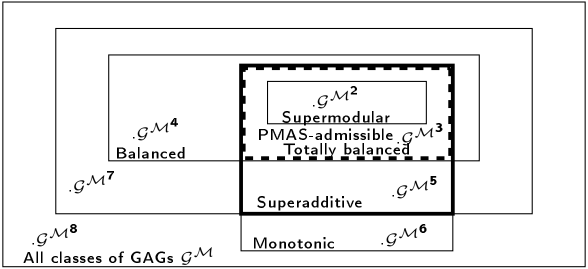

(b) corresponding classes of GAGs $\mathcal{G}^{\mathcal{M}}$

Weighted glove games

Glove games

Ingredients:

- a partition $\{L, R\}$ of the set of players N
- a weight vector $w \in \mathbb{R}^N_+$ (each player i in L owns w_i left gloves, each player j in R owns w_j right ones)
- a characteristic function $v(S) = \min\{\sum_{i \in S \cap L} w_i, \sum_{j \in S \cap R} w_j\}$ representing the profit obtained by members in S selling their pairs of gloves (sold at selling price of 1)
- Note that players are allowed to have a non-integer number of gloves
- If $w_i = 1$ for every $i \in N$ the game is a standard glove game.

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We can represent this game as a GAG $v_{\mathcal{M},w}$ by assigning by defining a coalitional map $\mathcal{M}$ such that for each coalition $S \in 2^N$:

$$\mathcal{M}(S) = \begin{cases} S \cap L & \text{if } \sum_{i \in S \cap L} w_i \leq \sum_{i \in S \cap R} w_i; \\ S \cap R & \text{otherwise.} \end{cases}$$

Proposition

Let (N, v) be a weighted glove game with positive weight vector w (so $w_i > 0$ for every $i \in N$) and let $\{L, R\}$ be the partition of N in 'left glove' and 'right glove' players. Then (N, v) is supermodular if and only if L contains precisely one player l^* and $w_{l^*} \geq \sum_{j \in R} w_j$ or R contains precisely one player r^* and $w_{r^*} \geq \sum_{i \in L} w_i$.

See
Moretti, S., Norde, H. (2021) A note on weighted multi-glove games. Soc Choice Welf. https://doi.org/10.1007/s00355-021-01337-8
for a generalisation of this results to weighted multi-glove games.

Using GAGs to prove the if part

$$\mathcal{M}(S) = \begin{cases} S \cap L & \text{if} \quad \sum_{i \in S \cap L} w_i \leq \sum_{j \in S \cap R} w_j \\ S \cap R & \text{otherwise,} \end{cases} \tag{1}$$

Suppose $\{L, R\}$ is a partition of the player set N with $|L| = 1$ (the case $|R| = 1$ can be treated in a similar way).

Let l^* be the unique element of L.

Observe that $\mathcal{M}(S) = S \cap R$ if $l^* \in S$ and $\mathcal{M}(S) = \emptyset$ otherwise.

It is straightforward to check that $\mathcal{M}$ is supermodular:
$\mathcal{M}(S) \cap \mathcal{M}(T) = \mathcal{M}(S \cap T)$

From costs to revenues

Given a *cost* game (N, c), with *cost function* $c : 2^N \to \mathbb{R}$, one can also consider the corresponding *cost saving game* (N, v^c) such that

$$v^c(S) = \sum_{i \in S} c(\{i\}) - c(S),$$

for each coalition $S \in 2^N$, where the difference $v^c(S)$ between the total cost in the situation where all members of S work alone and the cost in the situation where all members of S cooperate is interpreted as a profit of coalition S.

In alternative, one can also define the corresponding *dual game* (N, c^*) such that

$$c^*(S) = c(N) - c(N \setminus S),$$

for each coalition $S \in 2^N$, where the rest $c^*(S)$ obtained from the cost of the grand coalition N after the complement of coalition S pays its entire cost in the original game is also interpreted as a profit of coalition S.

Some well-known facts

Proposition

Let (N, c) be a cost game and let (N, v^c) be the corresponding cost saving game. Then the following statements hold true:

(i) *c is submodular iff v^c is supermodular;*

(ii) *c is subadditive iff v^c is superadditive;*

(iii) *c is (totally) balanced iff v^c is (totally) balanced;*

(iv) *c admits a PMAS iff v^c admits a PMAS.*

Proposition

Let (N, c) be a cost (profit) game and let its dual (N, c^*) be a profit (cost) game. Then the following statements hold true:

i) *c is monotonic iff c^* is monotonic;*

ii) *$C(c) = C(c^*)$ (c and c^* have the same core);*

iii) *c is submodular iff c^* is supermodular.*

Link connection games

Let $\Gamma = (N, E)$ be an undirected graph and $w \in \mathbb{N}^N$ a nonnegative integer weight vector. Edges will be denoted by ij instead of $\{i, j\}$

Definition

The *link connection game* associated with (V, E) and w is the cost game (E, c), such that

$$c(S) = \min\{W(T) \mid T \subseteq S \text{ and } \mathcal{P}_T = \mathcal{P}_S\} \tag{2}$$

for every $S \subseteq E$, where

- $W(T) = \sum_{e \in T} w_e$ and
- $\mathcal{P}_T$ denotes the set of all connected components in graph (V_S, T) for any $T \subseteq S$ (here, V_S is the set of all nodes of edges in S).

The corresponding cost saving game (E, v^c) is defined as follows:

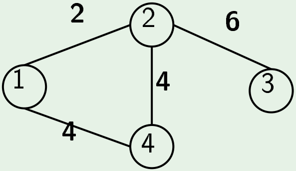
$$v^c(S) = \sum_{e \in S} w_e - c(S), \tag{3}$$

for every $S \subseteq E$

Link connection games

Example

A graph (V, E) with $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 14, 24\}$:



The corresponding link connection game game is

S	$\emptyset$	$\{12\}$	$\{23\}$	$\{14\}$	$\{24\}$	$\{12, 23\}$	$\{12, 14\}$	$\{12, 24\}$	$\{23, 14\}$	$\{23, 24\}$	$\{14, 24\}$
$c(S)$	0	2	6	4	4	8	6	6	10	10	8

S	$\{12, 23, 24\}$	$\{12, 23, 14\}$	$\{12, 14, 24\}$	$\{23, 14, 24\}$	E
$c(S)$	12	12	6	14	12

Coalitional map for cost saving game

- Start with listing the edges in E in some order $E = \{e_1, e_2, e_3, \ldots, e_{|E|}\}$ (not yet having a cost vector $w \in \mathbb{R}_+^E$ in mind).
- For every $S \subseteq E$ an edge $e_k \in S$, ($k \in \{1, \ldots, |E|\}$) is called *superfluous in S* if it forms a cycle with its predecessors in S, more precisely, if $(V, \{e_1, \ldots, e_{k-1}\} \cap S)$ and $(V, \{e_1, \ldots, e_k\} \cap S)$ have the same connected components.
- Selects the collection of superfluous edges in any coalition:

$$\mathcal{M}(S) = \{e \in S : e \text{ is superfluous in } S\}, \tag{4}$$

for every $S \subseteq E$.

Coalitional map for cost saving game

- Start with listing the edges in E in some order $E = \{e_1, e_2, e_3, \dots, e_{|E|}\}$ (not yet having a cost vector $w \in \mathbb{R}^E_+$ in mind).
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- Selects the collection of superfluous edges in any coalition:

$$\mathcal{M}(S) = \{e \in S : e \text{ is superfluous in } S\}, \tag{4}$$

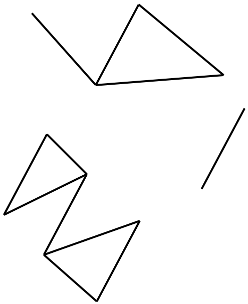
for every $S \subseteq E$.

Proposition

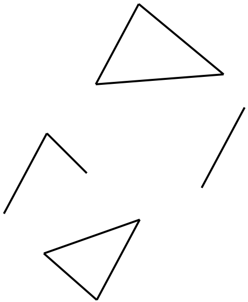
Let $\mathcal{M}$ be the coalitional map as defined in (4). Then $\mathcal{M}$ is monotonic, proper and veto-rich.

- Choose the ordering of the edges in E according to increasing costs, i.e. $E = \{e_1, e_2, e_3, \ldots, e_{|E|}\}$ such that $w_{e_1} \leq w_{e_2} \leq \cdots \leq w_{e_{|E|}}$
- Let $S \subseteq E$. The graph (V, S) partitions the vertex set V into components. Some components may be singletons, some may be trees and the other components are connected components containing cycles.
- In order to find a subset $T \subseteq S$ of minimal cost that results in the same partition of V into components we can use the well-known algorithm of Prim: reduce any component in (V, S) with a cycle to a tree by removing edges that form a cycle with the cheaper edges in S.
- Since we have chosen the order of E with respect to increasing costs this process boils down to removing the superfluous edges in S, i.e. removing the edges in $\mathcal{M}(S)$. So an optimal network for coalition S is $(V, S \backslash \mathcal{M}(S))$ and the cost saving, going from (V, S) to $(V, S \backslash \mathcal{M}(S))$, is equal to $\sum_{e \in \mathcal{M}(S)} w_e = v_{\mathcal{M},w}(S)$.
- As this is true for every $S \subseteq E$ we get $(E, v^c) = (E, v_{\mathcal{M},w})$.

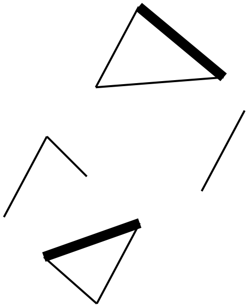
Consider a graph with set of edges E:



Take a coalition $S \subseteq E$



If edges are ordered in increasing way, superfluous edges are those who form a cycle with cheaper ones

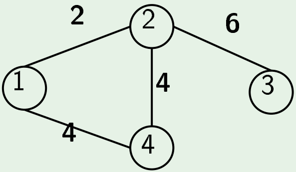


The sum of the weights of superfluous give the total saving for coalition S. So, $(E, v^c) = (E, v_{\mathcal{M},w})$.

Link connection games

Example

A graph (V, E) with $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 14, 24\}$:



The corresponding link connection game game is

S	$\emptyset$	$\{12\}$	$\{23\}$	$\{14\}$	$\{24\}$	$\{12,23\}$	$\{12,14\}$	$\{12,24\}$	$\{23,14\}$	$\{23,24\}$	$\{14,24\}$
$c(S)$	0	2	6	4	4	8	6	6	10	10	8
$\mathcal{M}(S)$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$v^c(S)$	0	0	0	0	0	0	0	0	0	0	0

S	$\{12,23,24\}$	$\{12,23,14\}$	$\{12,14,24\}$	$\{23,14,24\}$	E
$c(S)$	12	12	6	14	12
$\mathcal{M}(S)$	$\emptyset$	$\emptyset$	$\{24\}$	$\emptyset$	$\{24\}$
$v^c(S)$	0	0	4	0	4

Proposition

Let (V, E) be an undirected graph and $\mathcal{M}$ the coalitional map as defined in (4). Then $(E, v_{\mathcal{M},w})$ is monotonic, superadditive, (totally) balanced and PMAS-admissible for every $w \in \mathbb{R}_+^E$.

Proposition

Cost saving games corresponding to link connection games are monotonic, superadditive, (totally) balanced and PMAS-admissible.

Proposition

Link connection games are subadditive, (totally) balanced and PMAS-admissible.

Weighted minimum coloring games

Let $\Gamma = (N, E)$ be an undirected graph and $w \in \mathbb{N}^N$ a nonnegative integer weight vector.

A k-coloring of graph Γ wrt weight vector w is a function $h : N \to 2^{\{1,\dots,k\}}$ that assigns a set of w_i different colors to every vertex $i \in N$ such that adjacent vertices receive disjoint sets of colors and at most k colors are used ($|h(i)| = w_i$ for all $i \in N$ and $h(i) \cap h(j) = \emptyset$ for all $ij \in E$.

weighted chromatic number $\chi_w(\Gamma)$: the minimum number k such that a k-coloring of Γ with respect to w.

Definition

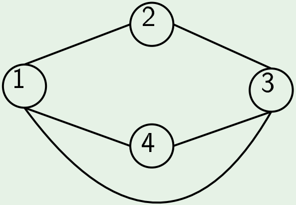
The weighted minimum coloring game (Hamers et al. 2019) on $\Gamma = (N, E)$ with weight vector $w \in \mathbb{N}^N$ is the cost game $(N, c^{\Gamma,w})$ defined by

$$c^{\Gamma,w}(S) = \chi_{w_S}(\Gamma_{|S})$$

for every $S \in 2^N$.

The unweighted case ($w_i = 1$ for all $i \in N$)

Example



S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$c(S)$	0	1	1	1	1	2	2	2	2	1	2
$c^*(S)$	0	1	0	1	0	2	2	1	1	1	1

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$c(S)$	3	2	3	2	3
$c^*(S)$	2	2	2	2	3

The unweighted case for complete multipartite graphs

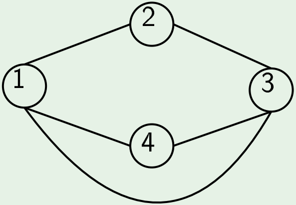
- A graph $G = (N, E)$ is complete multipartite if there is a partition $\{P_1, P_2, \ldots, P_r\}$ of the vertex set N such that for any two vertices $i \in P_k$, $j \in P_l$ we have $\{i, j\} \in E$ if and only if $k \neq l$.
- For every $k \in \{1, \ldots, r\}$ let p_k be the element of P_k with the smallest index. Define the coalitional map $\mathcal{M}$ by

$$\mathcal{M}(S) = \{p_k | k \in \{1, \ldots, r\}, P_k \subseteq S\} \tag{5}$$

- Then, the dual game $c^{\Gamma *}$ coincides with $v_{\mathcal{M},w}(S)$

The unweighted case ($w_i = 1$ for all $i \in N$)

Example



S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$c(S)$	0	1	1	1	1	2	2	2	2	1	2
$c^*(S)$	0	1	0	1	0	1	2	1	1	1	1
$\mathcal{M}(S)$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1\}$	$\{1,3\}$	$\{1\}$	$\{3\}$	$\{2\}$	$\{3\}$

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$c(S)$	3	2	3	2	3
$c^*(S)$	2	2	2	2	3
$\mathcal{M}(S)$	$\{1,3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,4\}$

Proposition

Let $\Gamma = (N, E)$ be a complete multipartite graph and let $\mathcal{M}$ be the coalitional map as defined in the previous slide. Then $\mathcal{M}$ is supermodular.

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Let $\Gamma = (N, E)$ *be a complete multipartite graph and let* $\mathcal{M}$ *be the coalitional map as defined in previous slide. Then* $(N, v_{\mathcal{M},w})$ *is convex for every* $w \in \mathbb{R}_+^N$.

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Matroids

A matroid is a pair $M = (E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subseteq 2^E$ such that

I) $\emptyset \in \mathcal{I}$;

II) if $T \in \mathcal{I}$ and $S \subseteq T$, then $S \in \mathcal{I}$ (*independent set*);

III) if $S, T \in \mathcal{I}$ with $|S| < |T|$, then there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{I}$.

As an example of matroid, consider the graphic matroid $M_G = (E_G, \mathcal{I}_G)$ on a graph (V, E)

- The set E_G is defined to be E, the set of edges of graph G.
- A subset $A \subseteq E$ is an independent set ($A \in \mathcal{I}_G$) if and only if the subgraph $G_A = (V_A, A)$ forms a forest.

Minimum base games on matroids

- We can add a vector of weights to the elements of E of a matroid $M = (E, \mathcal{I})$, so we have a weighted matroid.
- In Nagamochi, H., Zeng, D. Z., Kabutoya, N., Ibaraki, T. (1997) Complexity of the minimum base game on matroids. Mathematics of Operations Research, 22(1), 146-164.
- The authors consider a minimum base game on a weighted matroid where the cost of each coalition $S \subseteq E$ is the total weight of a minimum base on S, where a base on S is defined as a maximal (wrt inclusion) subset of S that is also independent set.
- In the case of a graphic matroid, a minimum base game is a link connection games.

- Nagamochi et al. (1997) have shown that a minimum base game has a nonempty core if and only if the weighted matroid has no all-negative circuits.
- Q.: What about PMAS?
- Work in progress:
  - we generalize the coalitional map for link connection games to weighted matroids
  - so minimum base games are GAGs (note that weights can be negative, but "superfluous" are positive under the condition of no all-negative circuits)
  - using the machinery of GAGs we can prove that minimum base games on weighted matroids with no all-negative circuits are subadditive, (totally) balanced and PMAS-admissible
  - and the way around using the result in Nagamochi et al. (1997)

Thank you for your attention

- Moretti, S., Norde, H. (2021) Some new results on generalized additive games. Int J Game Theory. https://doi.org/10.1007/s00182-021-00786-w
- Moretti, S., Norde, H. (2021) A note on weighted multi-glove games. Soc Choice Welf. https://doi.org/10.1007/s00355-021-01337-8