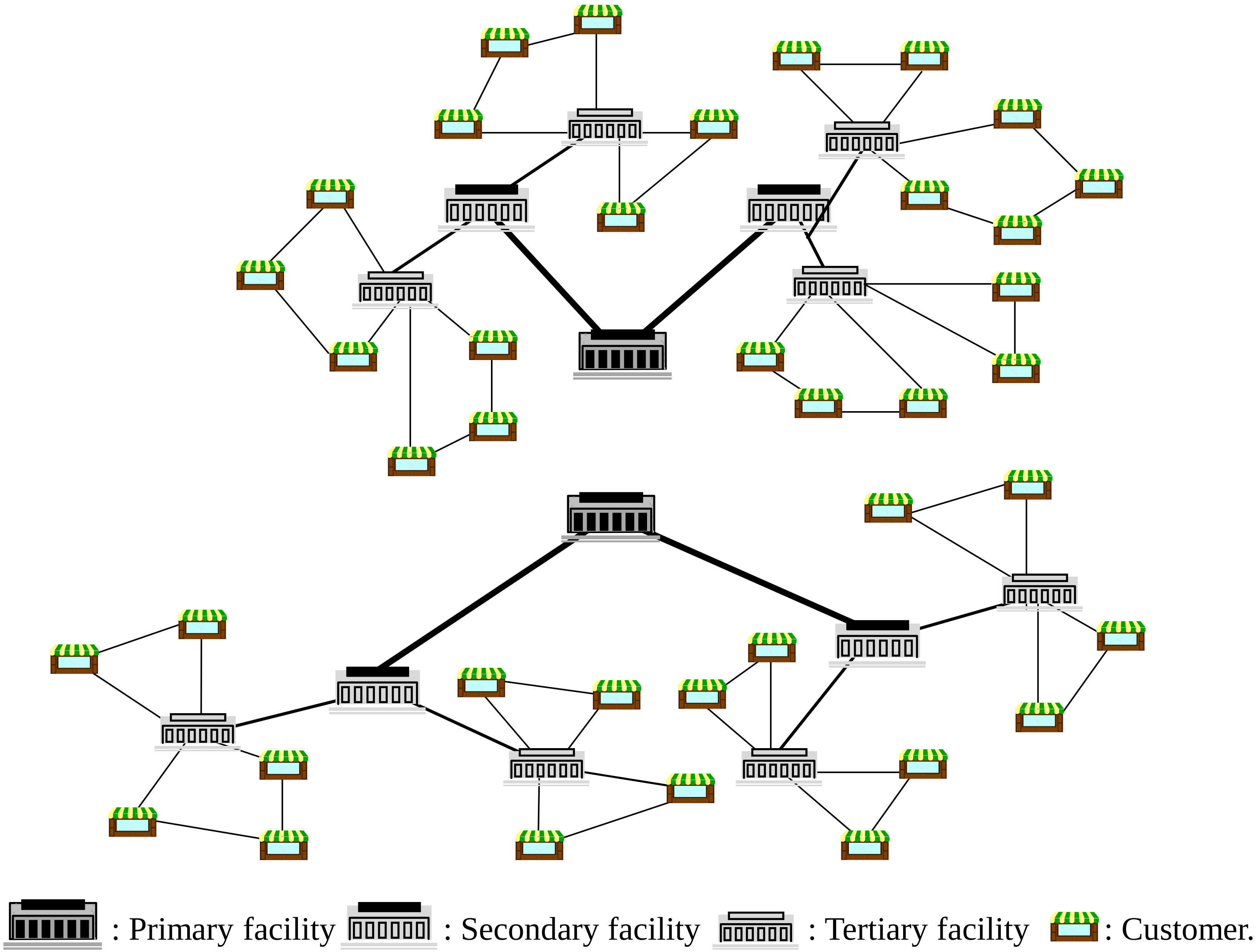
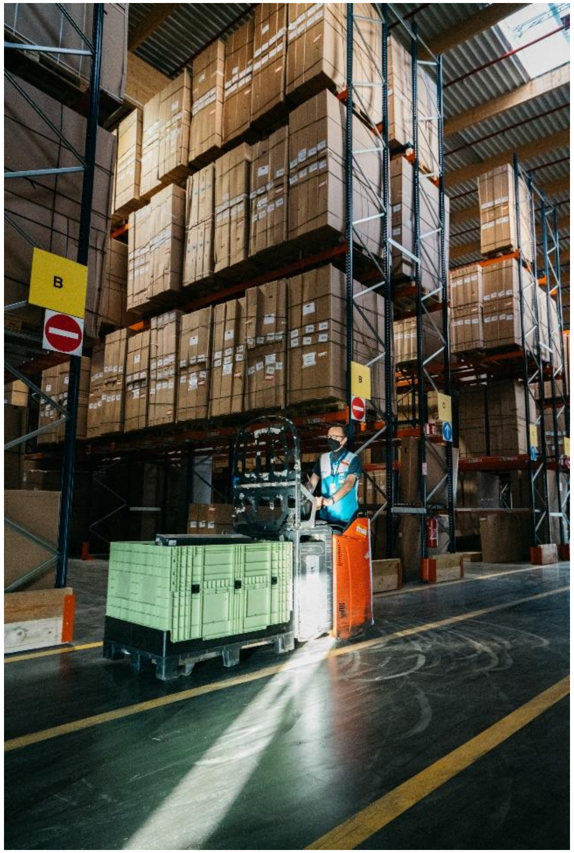
Models and Methods
for Two-Level Facility Location Problems

Frédéric SEMET, Bernard GENDRON, Paul-Virak Khuong


: Primary facility
: Secondary facility
: Tertiary facility
: Customer.

Two real-life examples

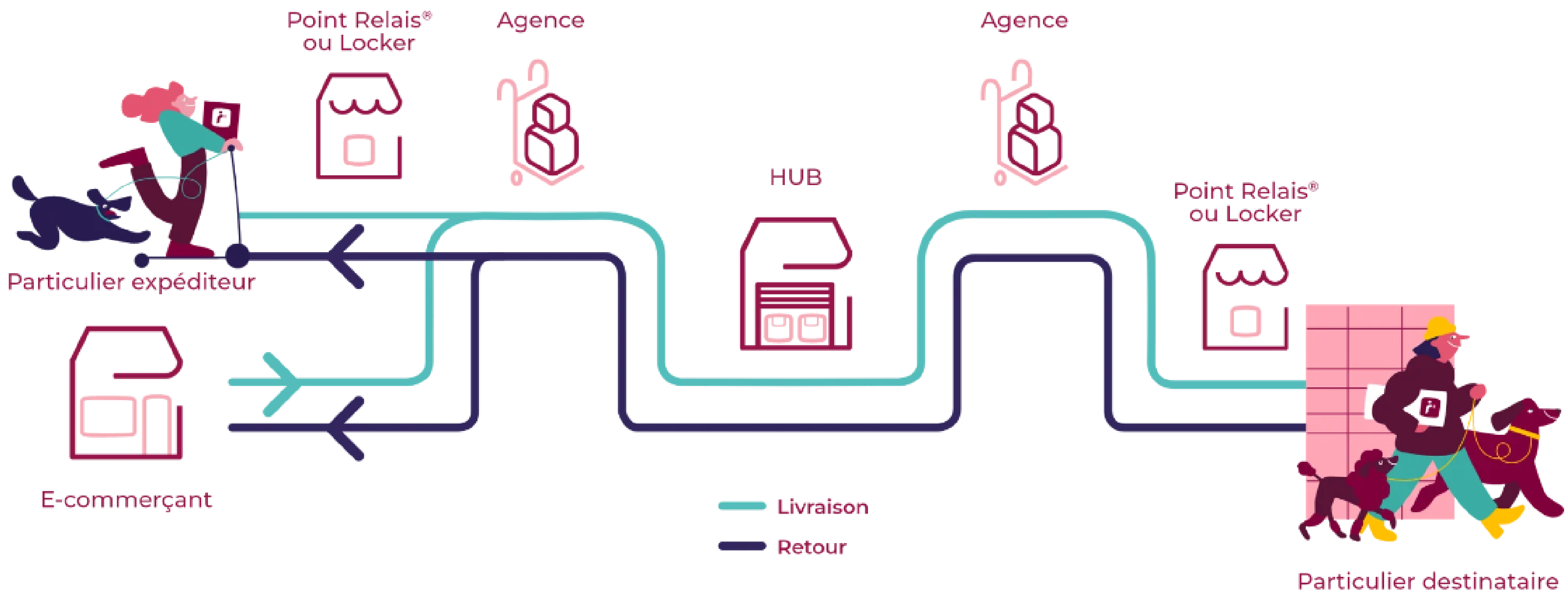
Adapted from Voix du Nord, 2013



Decathlon, 2024

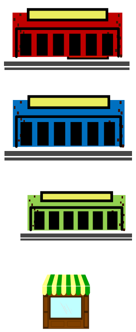
Two real-life examples

Mondial Relay



Mondial Relay, 2024

Motivation : e-commerce distribution system

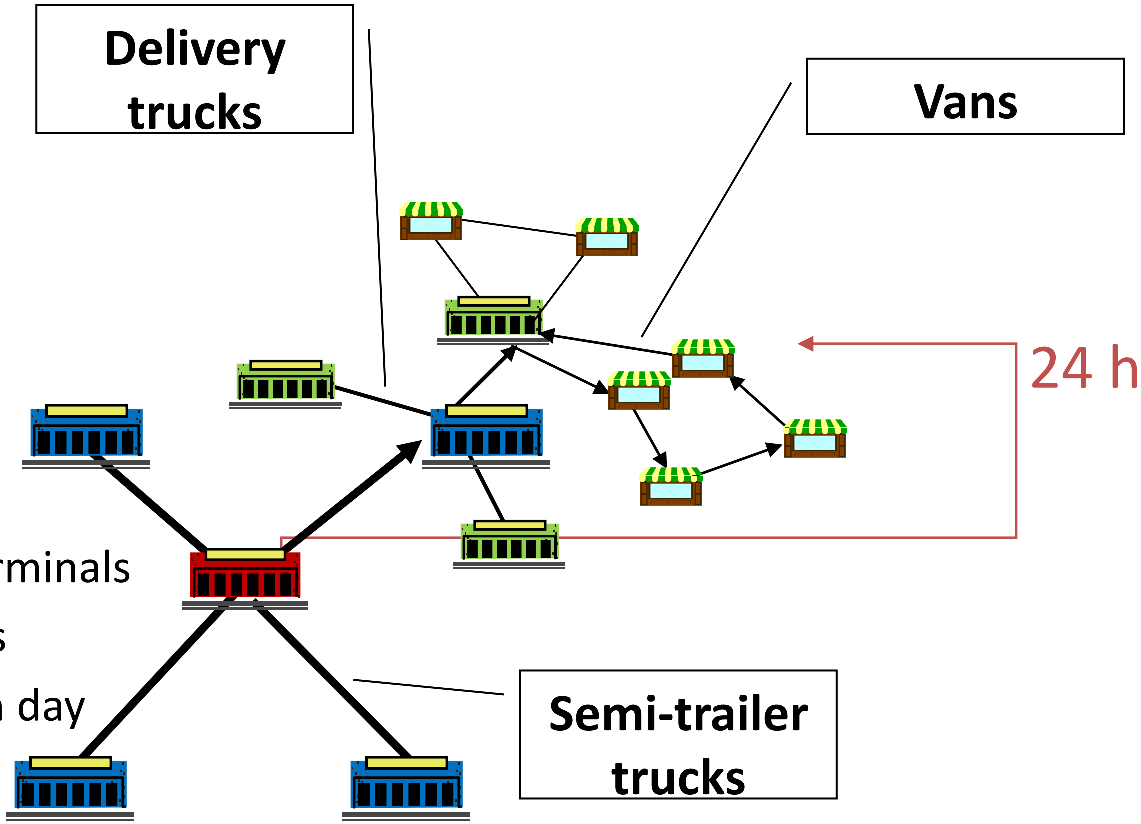


: Warehouses

: Cross-dock terminals

: Satellites

: Pick-up & Go



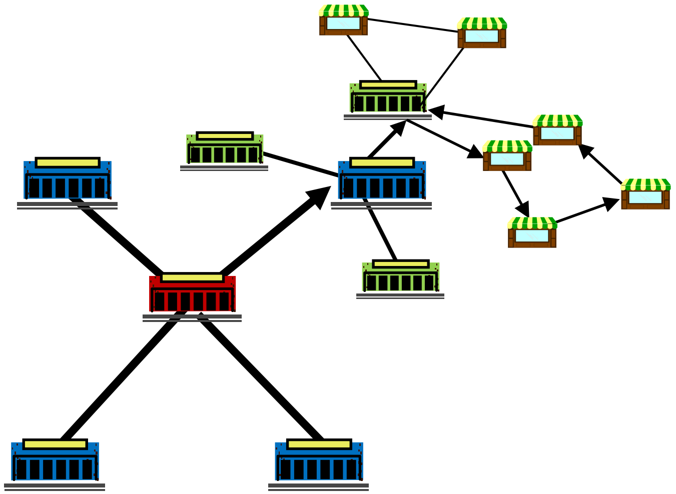
- 2 Warehouses
- 93 potential cross-dock terminals
- 320 potential satellite sites
- 8800 delivery points each day

Determine warehouse and satellite locations as well as delivery routes in a such way that the total logistics cost of the system is minimized

Pre-processing phase: Location part

Warehouse locations given :

- No fixed cost
- All types of goods available



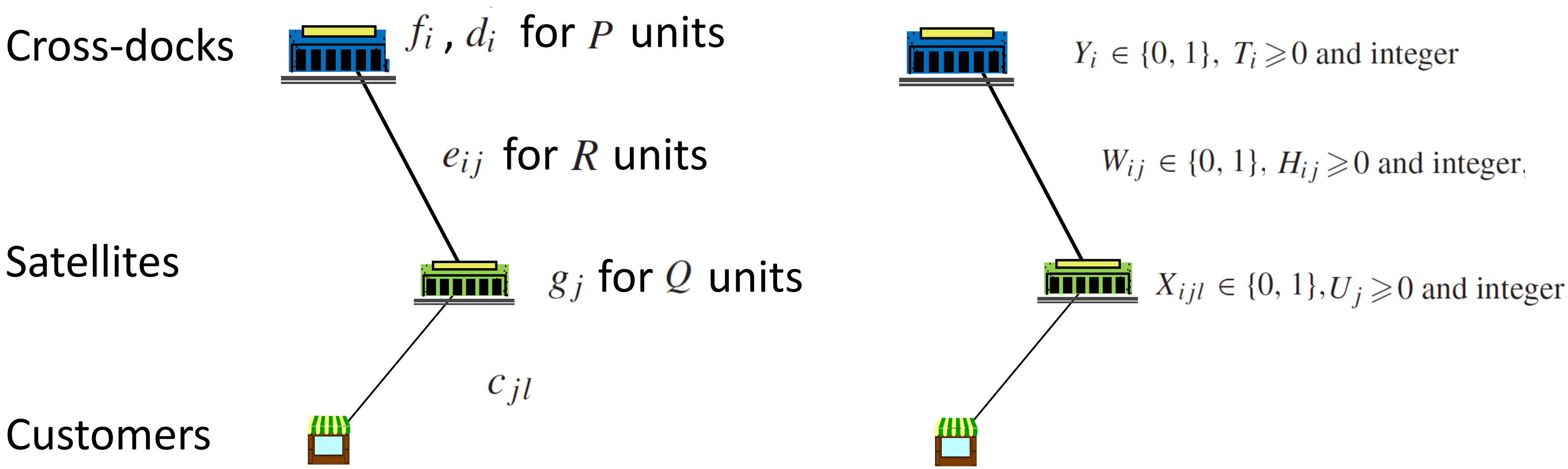
→ *Each potential cross-dock is served from the closest warehouse*

- fixed costs

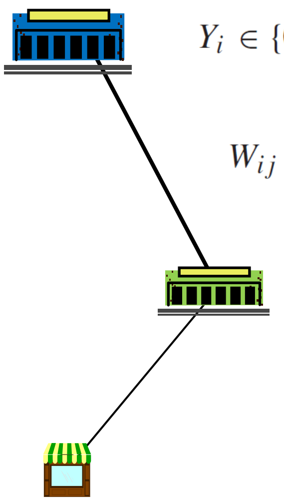
- variable costs (number of vehicles)

→ *Two-echelon capacitated location-distribution problem*

Path-based model (Gendron, Semet (2009))



Path-based model (Gendron, Semet (2009))



$Y_i \in \{0, 1\},\ T_i \geqslant 0$ and integer

$W_{ij} \in \{0, 1\},\ H_{ij} \geqslant 0$ and integer.

$X_{ijl} \in \{0, 1\}, U_j \geqslant 0$ and integer

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} g_j U_j + \sum_{i \in D} d_i T_i + \sum_{i \in D} \sum_{j \in S_i^D} e_{ij} H_{ij} + \sum_{i \in D} \sum_{j \in S_i^D} \sum_{l \in L_j^S} c_{jl} X_{ijl},$$

$$\sum_{j \in S_l^L} \sum_{i \in D_j^S} X_{ijl} = 1, \quad \forall l \in L,$$

$$\sum_{i \in D_j^S} W_{ij} \leqslant 1, \quad \forall j \in S,$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leqslant \left(\sum_{l \in L_i^D} v_l \right) Y_i, \quad \forall i \in D,$$

$$X_{ijl} \leqslant W_{ij}, \quad \forall i \in D, \ \forall j \in S_i^D, \ \forall l \in L_j^S,$$

$$W_{ij} \leqslant Y_i, \quad \forall i \in D, \ \forall j \in S_i^D,$$

$$\sum_{i \in D_j^S} \sum_{l \in L_j^S} n_l X_{ijl} \leqslant Q U_j, \quad \forall j \in S,$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leqslant P T_i, \quad \forall i \in D,$$

$$\sum_{l \in L_j^S} v_l X_{ijl} \leqslant R H_{ij}, \quad \forall i \in D, \ \forall j \in S_i^D,$$

Main results (Gendron, Semet (2009))

Reformulation of *BIN*(M_{path}):

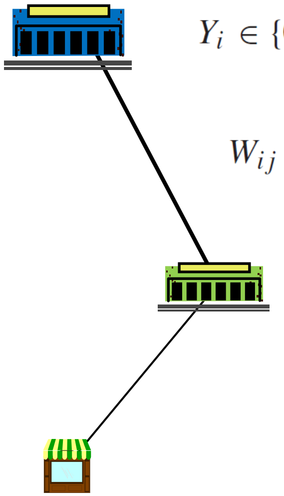
$BIN(M_{path})$ is equivalent to M_{splp}

Bounds Relationships:

$$Z(M_{arc}) = Z(M_{path}) \geq$$
$$Z(BIN(M_{arc})) = Z(BIN(M_{path})) = Z(M_{splp})$$
$$\geq Z(LP(M_{splp})) \geq Z(LP(M_{path})) \geq Z(LP(M_{arc}))$$

A solution for the distribution network

Path-based model (Gendron, Semet (2009))



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$$\sum_{j \in S_l^L} \sum_{i \in D_j^S} X_{ijl} = 1, \quad \forall l \in L,$$

$$\sum_{i \in D_j^S} W_{ij} \leqslant 1, \quad \forall j \in S,$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leqslant \left(\sum_{l \in L_i^D} v_l \right) Y_i, \quad \forall i \in D,$$

$$X_{ijl} \leqslant W_{ij}, \quad \forall i \in D, \ \forall j \in S_i^D, \ \forall l \in L_j^S,$$

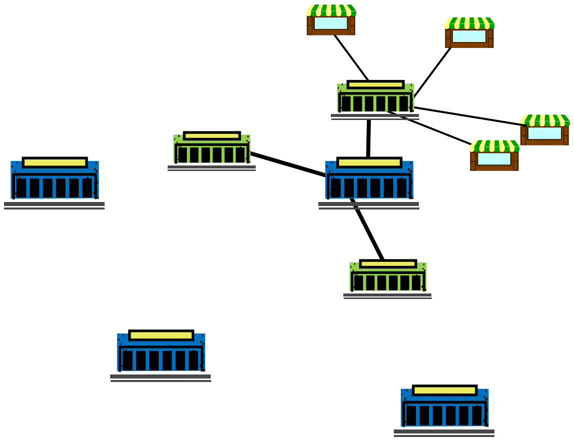
$$W_{ij} \leqslant Y_i, \quad \forall i \in D, \ \forall j \in S_i^D,$$

$$\sum_{i \in D_j^S} \sum_{l \in L_j^S} n_l X_{ijl} \leqslant Q U_j, \quad \forall j \in S,$$

$$\sum_{j \in S_i^D} \sum_{l \in L_j^S} v_l X_{ijl} \leqslant P T_i, \quad \forall i \in D,$$

$$\sum_{l \in L_j^S} v_l X_{ijl} \leqslant R H_{ij}, \quad \forall i \in D, \ \forall j \in S_i^D,$$

Two-level uncapacity facility location problem



Given two sets of locations (depots and satellites) and a set of customers

Select subsets of depots and satellites such that
the path to each customer begins at a depot and transits by a satellite
to minimize an objective function.

The objective function includes fixed costs associated with the depots and the satellites, transportation costs between depots and satellites, and from any depot to each customer.

TUFLP with single assignment constraints

The TUFLP-S with single assignment constraints imposes the additional restriction that each satellite can be connected to at most one depot.

Applications:
in transportation (Tragantalerngsak et al. 1997)
in telecommunications (Chardaire et al., 1999).

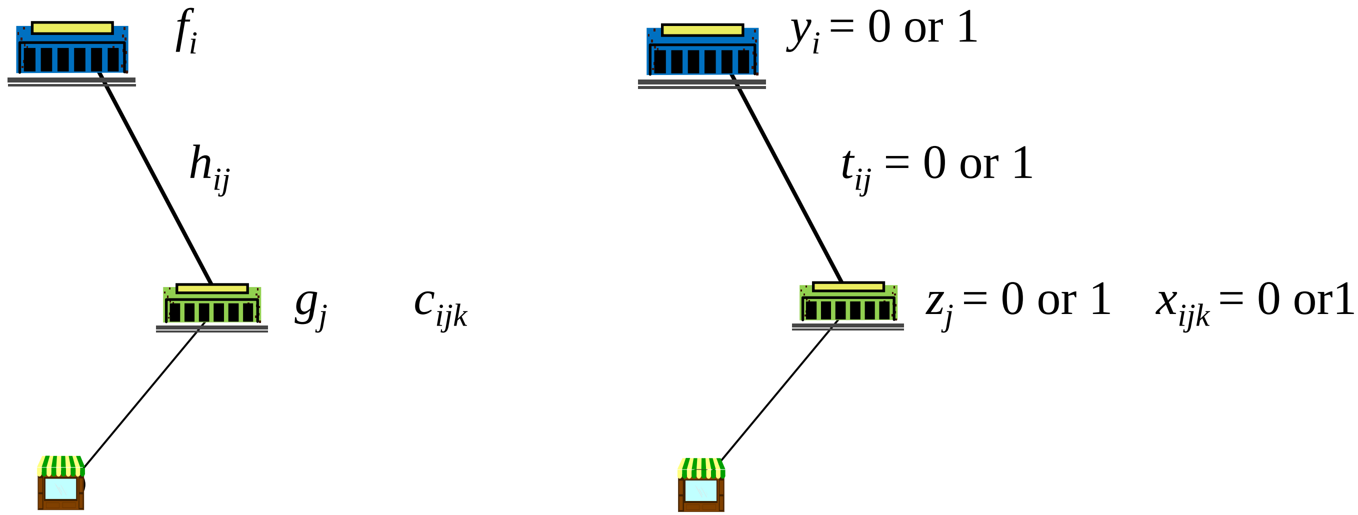
It exists a large class of TUFLP instances for which the single assignment constraints are not explicitly enforced, and there is an optimal solution that satisfies these constraints, due to the structure of the objective function.

Notation and Variable definitions

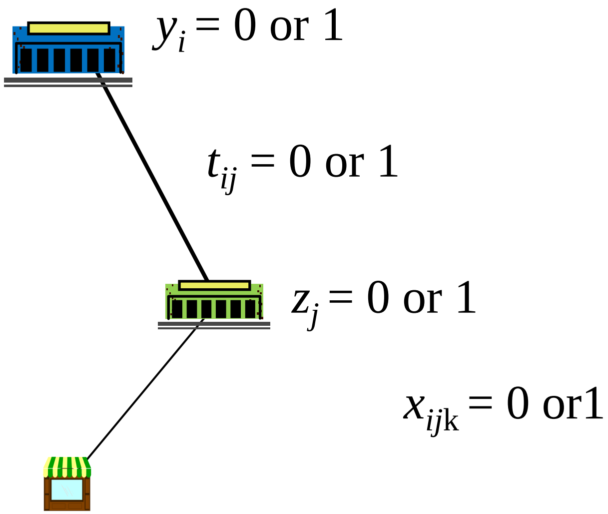
Depots : $i \in I$

Satellites : $j \in J$

Customers : $k \in K$



TUFLP formulation (G) (Barros and Labbé, 1999)



$$\min_{y,z,t,x} \sum_{i\in I} f_i y_i + \sum_{j\in J} g_j z_j + \sum_{(i,j)\in I\times J} h_{ij} t_{ij} + \sum_{(i,j,k)\in I\times J\times K} c_{ijk} x_{ijk},$$

subject to

$$\sum_{(i,j)\in I\times J} x_{ijk} = 1, \quad \forall k \in K,$$

$$x_{ijk} \le t_{ij}, \quad \forall (i,j,k) \in I\times J\times K,$$

$$\sum_{j\in J} x_{ijk} \le y_i, \quad \forall (i,k) \in I\times K,$$

$$\sum_{i\in I} x_{ijk} \le z_j, \quad \forall (j,k) \in J\times K,$$

$$0 \le x_{ijk} \le 1, \quad \forall (i,j,k) \in I\times J\times K,$$

$$y_i \in \{0,1\}, \quad \forall i \in I,$$

$$z_j \in \{0,1\}, \quad \forall j \in J,$$

$$t_{ij} \in \{0,1\}, \quad \forall (i,j) \in I\times J.$$

TUFLP-S weak formulation (W) (Gendron et al., 2017)

We add constraints enforcing the single assignment of satellites to depots:

$$\sum_{i \in I} t_{ij} \leq 1, \quad \forall j \in J.$$

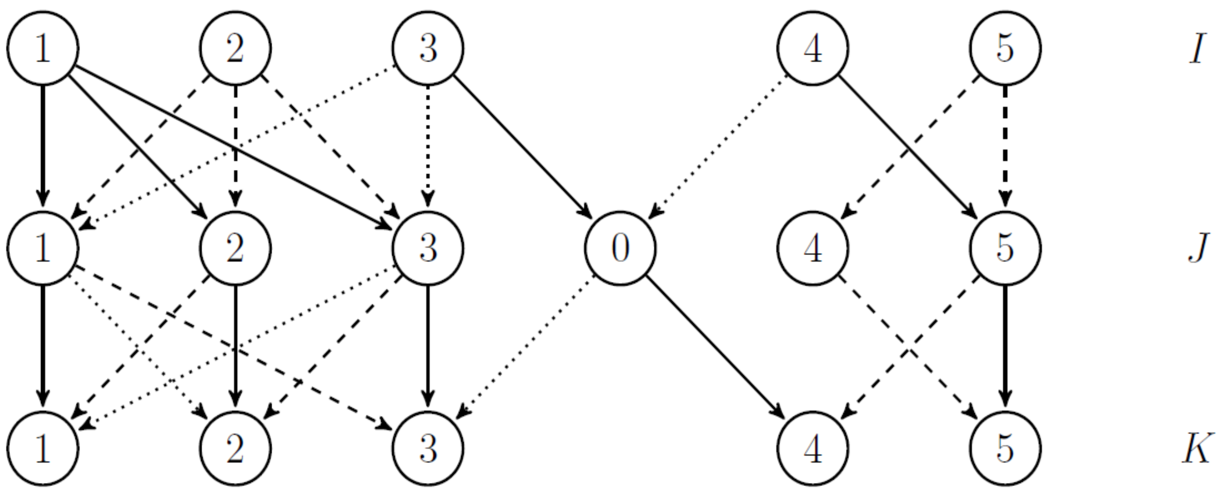
$f_i = 1$

$h_{ij} = 0$

$g_0 = \varepsilon < 1$

$g_j = 1, j = 1, \ldots, 5$

$c_{ijk} = 0$



We have : $v(G) = 5 + \varepsilon < v(W) = 7$

TUFLP-S strong formulation (S)) (Gendron et al., 2017)

$$\min_{y,z,t,x} \sum_{i\in I} f_i y_i + \sum_{j\in J} g_j z_j + \sum_{(i,j)\in I\times J} h_{ij} t_{ij} + \sum_{(i,j,k)\in I\times J\times K} c_{ijk} x_{ijk},$$

subject to

$$\sum_{(i,j)\in I\times J} x_{ijk} = 1, \qquad \forall k \in K,$$

$$x_{ijk} \le t_{ij}, \qquad \forall (i,j,k) \in I \times J \times K,$$

$$\sum_{j\in J} x_{ijk} \le y_i, \qquad \forall (i,k) \in I \times K,$$

$$\sout{\sum_{i\in I} x_{ijk} \le z_j, \qquad \forall (j,k) \in J \times K,}$$

$$\sum_{i\in I} t_{ij} = z_j, \qquad \forall j \in J.$$

Proposition 2:

$v_{LP}(W) \le v_{LP}(S)$ and the inequality can be strict.

TUFLP-S strong formulation simplified (S_P) (Gendron et al., 2017)

Using $\sum_{i \in I} t_{ij} = z_j,$:

$$\min_{y,t,x} \sum_{i \in I} f_i y_i + \sum_{(i,j) \in I \times J} (g_j + h_{ij}) t_{ij} + \sum_{(i,j,k) \in I \times J \times K} c_{ijk} x_{ijk}$$

subject to

$$\sum_{(i,j) \in I \times J} x_{ijk} = 1, \qquad \forall k \in K,$$

$$x_{ijk} \leq t_{ij}, \qquad \forall (i,j,k) \in I \times J \times K,$$

$$\sum_{j \in J} x_{ijk} \leq y_i, \qquad \forall (i,k) \in I \times K,$$

$$\forall i \in J.$$

$$0 \leq x_{ijk} \leq 1, \qquad \forall (i,j,k) \in I \times J \times K,$$

$$y_i \in \{0,1\}, \qquad \forall i \in I,$$

$$t_{ij} \in \{0,1\}, \qquad \forall (i,j) \in I \times J.$$

TUFLP-S formulations (Gendron et al., 2017)

By introducing constraints: $t_{ij} \leq y_i, \quad \forall (i,j) \in I \times J.$

Only non negativity has to be imposed on y_i variables

→ Models (W^C), (S^C), ($S_P{}^C$) in which integrality requirements are only imposed on t_{ij} variables

Proposition 3:

$$v_{LP}(W) = v_{LP}(W^C) \leq v_{LP}(S) = v_{LP}(S_P) = v_{LP}(S^C) = v_{LP}(S_P{}^C)$$

TUFLP-S formulation (Gendron et al., 2016)

$$\min \sum_{i \in I} f_i y_i + \sum_{(i,j) \in I \times J} l_{ij} t_{ij} + \sum_{(i,j,k) \in I \times J \times K} c_{ijk} x_{ijk}$$

subject to

$$\sum_{(i,j) \in I \times J} x_{ijk} = 1, \quad \forall k \in K,$$

$$\sum_{i \in I} t_{ij} \leq 1, \quad \forall j \in J,$$

$$\boxed{x_{ijk} \leq t_{ij},} \quad \forall (i,j,k) \in I \times J \times K,$$

$$\sum_{j \in J} x_{ijk} \leq y_i, \quad \forall (i,k) \in I \times K,$$

$$t_{ij} \leq y_i, \quad \forall (i,j) \in I \times J,$$

$$0 \leq x_{ijk} \leq 1, \quad \forall (i,j,k) \in I \times J \times K,$$

$$0 \leq y_i \leq 1, \quad \forall i \in I,$$

$$t_{ij} \in \{0,1\}, \quad \forall (i,j) \in I \times J.$$

Lagrangian relaxation of the TUFLP-S formulation (Gendron et al., 2016)

$$\min \sum_{i \in I} f_i y_i + \sum_{(i,j) \in I \times J} \left(l_{ij} - \sum_{k \in K} \lambda_{ijk} \right) t_{ij} + \sum_{(i,j,k) \in I \times J \times K} (c_{ijk} + \lambda_{ijk}) x_{ijk}$$

subject to

$$\sum_{(i,j) \in I \times J} x_{ijk} = 1, \qquad \forall k \in K,$$

$$\sum_{i \in I} t_{ij} \leq 1, \qquad \forall j \in J,$$

$$\sum_{j \in J} x_{ijk} \leq y_i, \qquad \forall (i,k) \in I \times K,$$

$$t_{ij} \leq y_i, \qquad \forall (i,j) \in I \times J,$$

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$$0 \leq t_{ij} \leq 1, \qquad \forall (i,j) \in I \times J,$$

$$y_i \in \{0,1\}, \qquad \forall i \in I.$$

Lagrangian subproblem (Gendron et al., 2016)

$$\min \sum_{i \in I} f_i y_i + \sum_{(i,j) \in I \times J} \left(l_{ij} - \sum_{k \in K} \lambda_{ijk} \right) t_{ij} + \sum_{(i,k) \in I \times K} \tilde{c}(\lambda)_{ik} w_{ik}$$

subject to

$$\sum_{i \in I} w_{ik} = 1, \qquad \forall k \in K,$$

$$\sum_{i \in I} t_{ij} \leq 1, \qquad \forall j \in J,$$

$$w_{ik} \leq y_i, \qquad \forall (i,k) \in I \times K,$$

$$t_{ij} \leq y_i, \qquad \forall (i,j) \in I \times J,$$

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$$0 \leq t_{ij} \leq 1, \qquad \forall (i,j) \in I \times J,$$

$$y_i \in \{0,1\}, \qquad \forall i \in I.$$

Uncapacitated facility location problem

Lagrangian-based Branch-and-Bound (Gendron et al., 2016)

Primal heuristic:

Invoke alternatively two large neighborhoods:

- Set of open depots fixed
- Assignment of customers to satellites

→ Solve Uncapacitated Location Problems

Branching rules:

- Branch on $\sum_{i \in I} t_{ij}$
- Branch on t_{ij}

Variable selection based on an adaptation of reliability branching

Some computational results …. (Gendron et al., 2016)

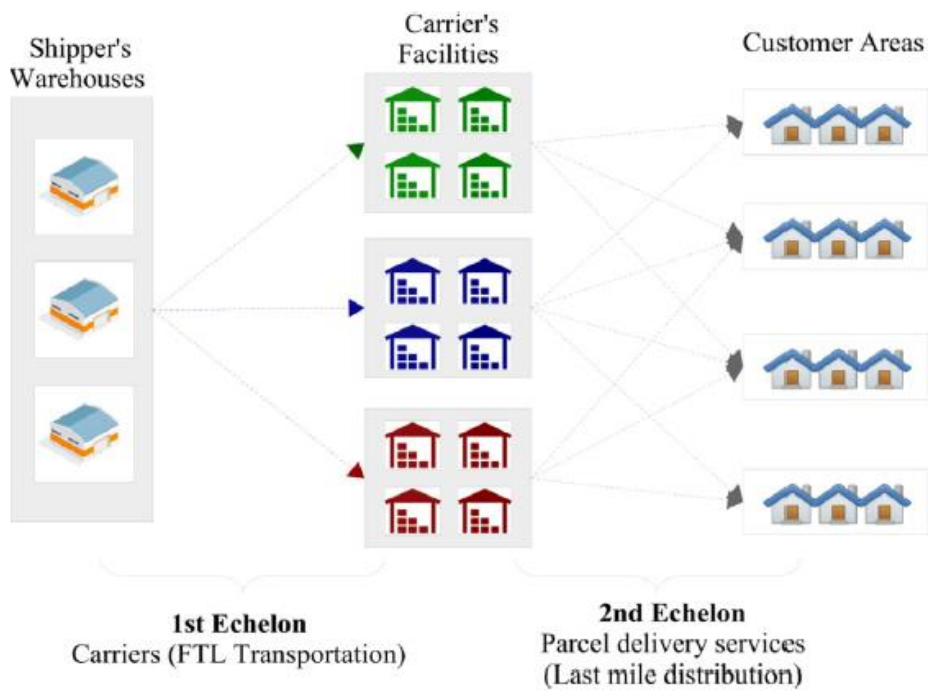
Instances	MIP-S	Lag/Pol	Lag/GUB
Tiny			
Time (s)	0.08	0.16	0.16
Nodes	1.03	0.49	0.56
Small			
Time (s)	4.95	239.07	4,021.88
Nodes	56.89	77.53	582.39
Medium			
Time (s)	23.97	115.13	1,176.68
Nodes	37.18	4.42	81.68
Full			
Time (s)	321.71	1,036.06	1,606.06
Nodes	162.51	11.46	36.50

Industrial instances

Instances	MIP-S	Lag/GUB
Large A-S		
Time (s)	3,622.68	1,293.26
Nodes	38,163.90	798.14
Large B-S		
Time (s)	4,009.84	1,203.34
Nodes	41,169.53	660.47
Large C-S		
Time (s)	4,892.40	1,593.91
Nodes	38,857.73	760.80

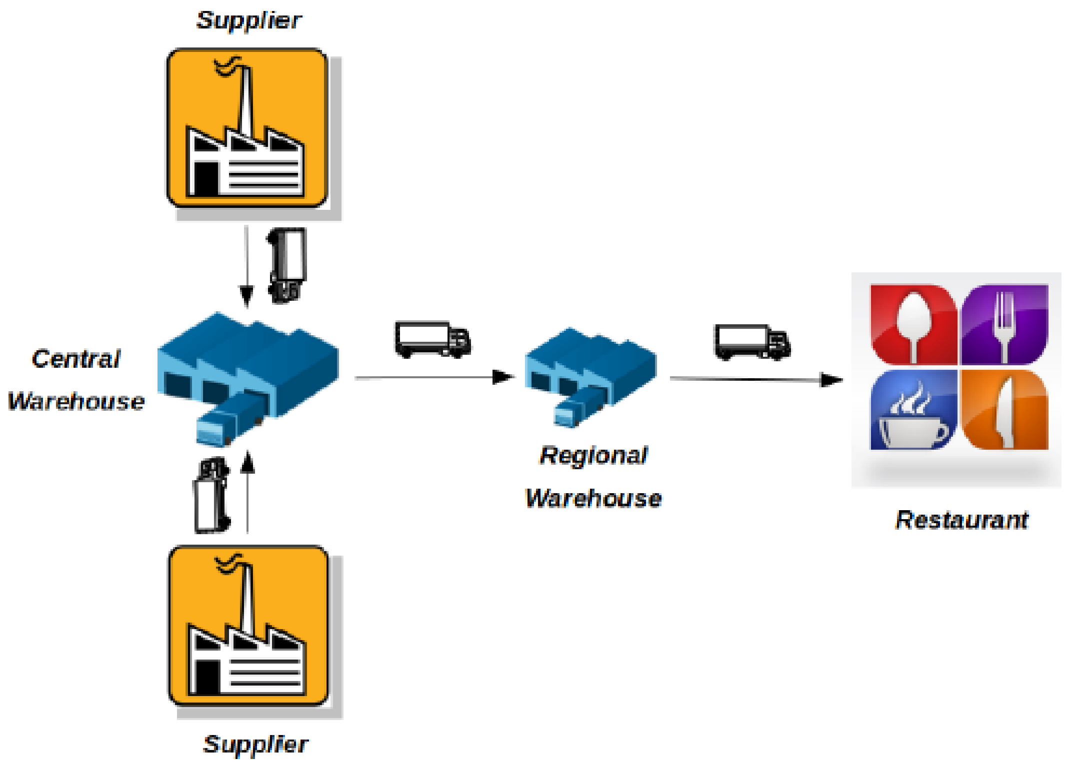
Large randomly generated instances

Conclusions



Multi-period distribution networks with purchase commitment contracts (Clavijo et al., 2024)

Conclusions



Logistics service network design (Belières et al., 2020, 2021, 2022)

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