A Uniformization Approach for the Dynamic Control of Queueing Systems with Abandonments

Benjamin Legros; Oualid Jouini; Ger Koole

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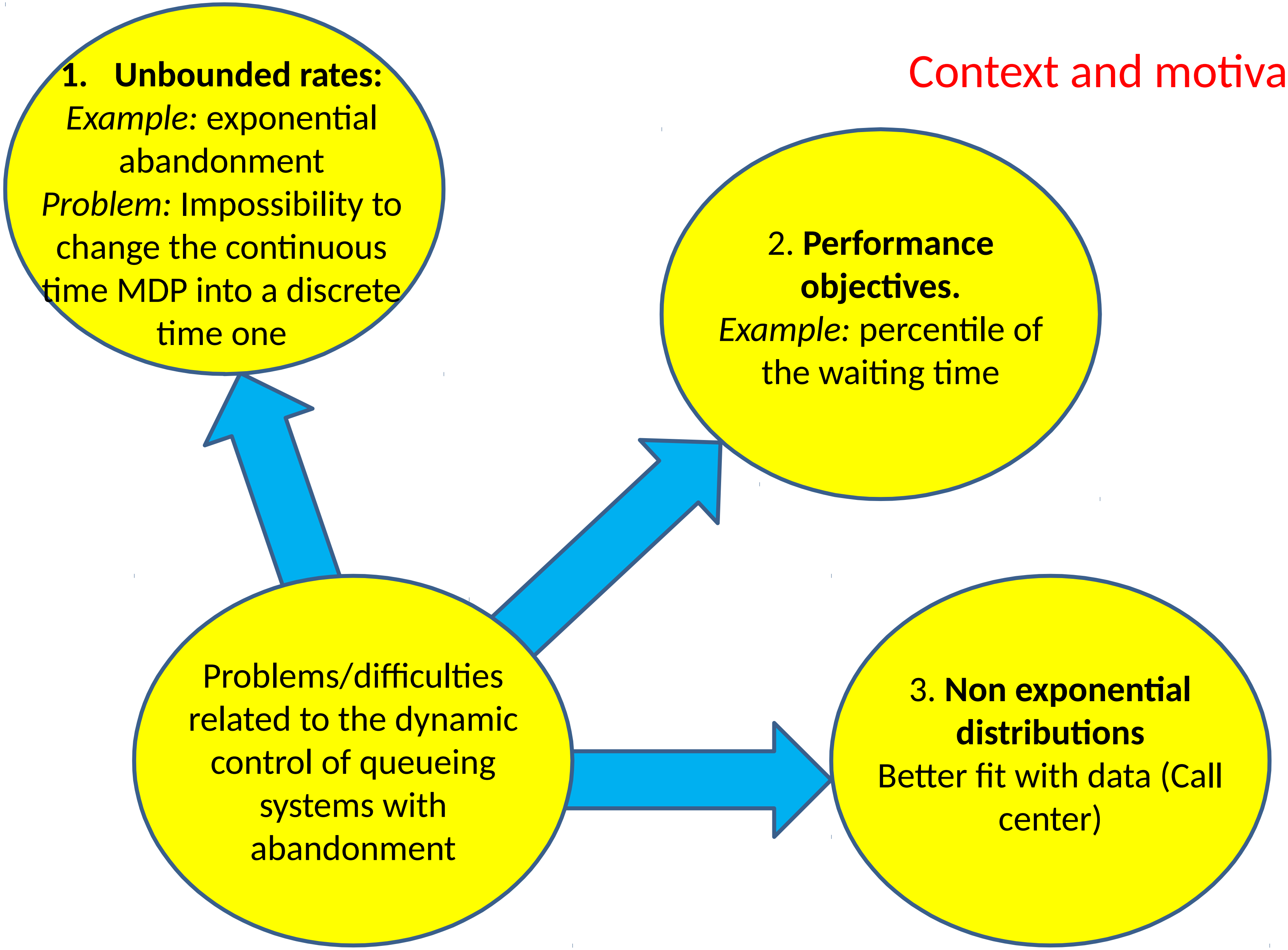
Context and motivations



Elements from the Literature

Direct Truncation

Problem: Break in the monotonicity properties. Example in an M/M/1+M queue

1. Unbounded rates

Solution (Bhulai): Replace λ by $\lambda\left(1-\frac{x}{N}\right)$

Bhulai S, Brooms AC, Spieksma FM (2014) On structural properties of the value function for an unbounded jump markov process with an application to a processor sharing retrial queue. *Queueing Systems* 76(4):425–446.

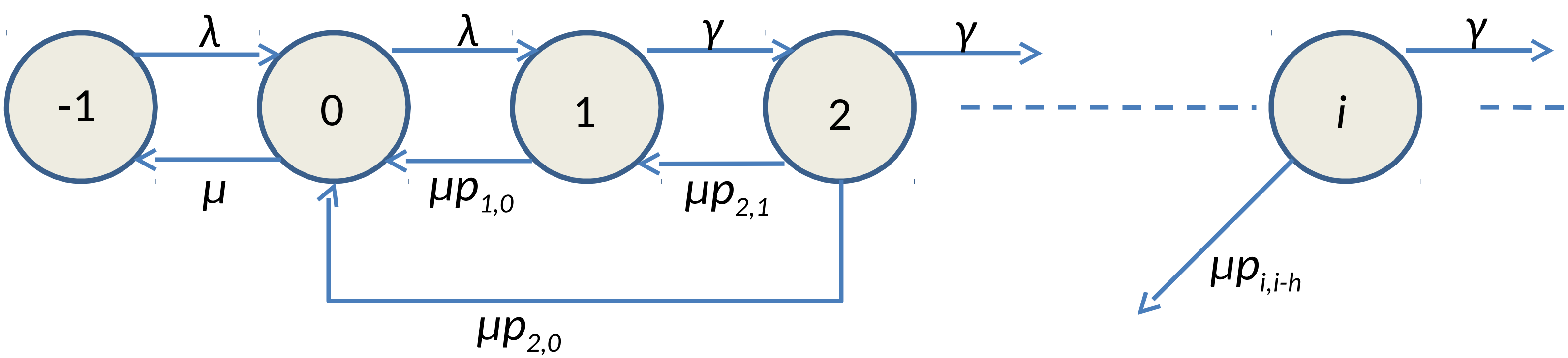
The Erlang Approximation for the waiting time of the First in Line (FIL) (Koole et Al. (2012))

Koole G, Nielsen BF, Nielsen TB (2012) First in line waiting times as a tool for analysing queueing systems. *Operations research* 60(5):1258–1266.

Elements from the Literature

2. Performance objectives

M/M/1 queue with FIL



$$p_{i,i-h} = \begin{cases} 1 - \sum_{h=0}^{i-1} \left(\frac{\lambda}{\lambda+\gamma}\right)\left(\frac{\gamma}{\lambda+\gamma}\right)^h & for\ i = h \\ \left(\frac{\lambda}{\lambda+\gamma}\right)\left(\frac{\gamma}{\lambda+\gamma}\right)^h, & for\ 0 \le h < i. \end{cases}$$

Contributions

Use of distribution which are dense.
Example: Coxian distribution
Schasberger R (1973) *Warteschlangen* (Springer-Verlag Vienna).

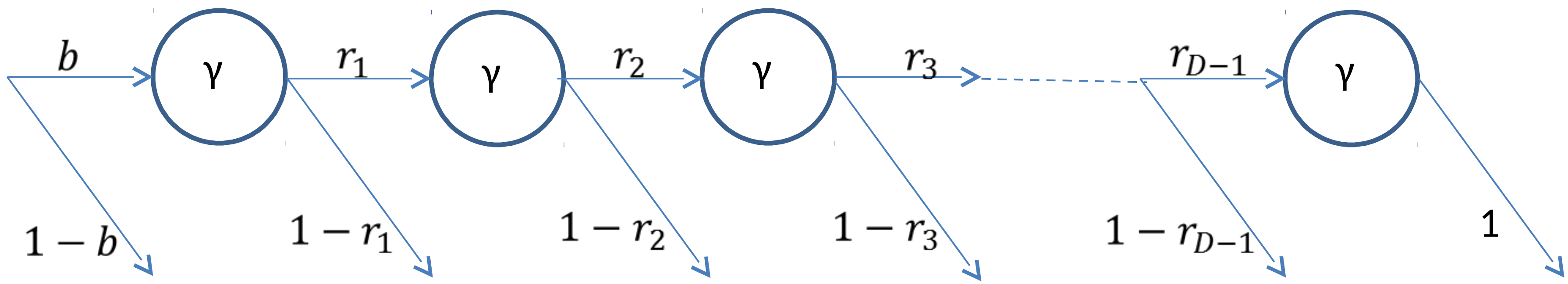
3. Non exponential distributions

Contribution

1. **Extension** of the FIL model with general abandonment.
2. **Proposition**: Approximate the abandonment distribution by a homogeneous Cox distribution
3. **Results:**
   a. Density of homogeneous Cox distribution
   b. Explicit Modeling of the FIL through a bounded jump Markov process
4. **Applicability:**
   a. Solving optimization problems (dynamic programming)
   b. Performance evaluation (Markov chain analysis)

Discretization of the First in Line Waiting Time

The homogeneous Coxian distribution: $X_{\gamma,D}$ (particularity: same rate as the elapsing of time)



THEOREM 1. *Let X be a non-negative random variable. There exists parameters of the homogeneous γ-Cox random variable, $X_{\gamma,D}$, such that $X_{\gamma,D}$ converges in distribution to X in the sense*

$$\lim_{\gamma\to\infty}\left(\lim_{D\to\infty} P(X_{\gamma,D} < t)\right) = P(X < t),$$

for any $t \geq 0$.

Steps of the proof:

1. Existence and uniqueness of $P(X_\gamma < t) = \lim_{D\to\infty} P\left(X_{\gamma,D} < t\right)$
2. We prove that the homogeneous Coxian random variable can arbitrarily closely approximate in distribution any Cox random variable
3. Coxian random variables are dense

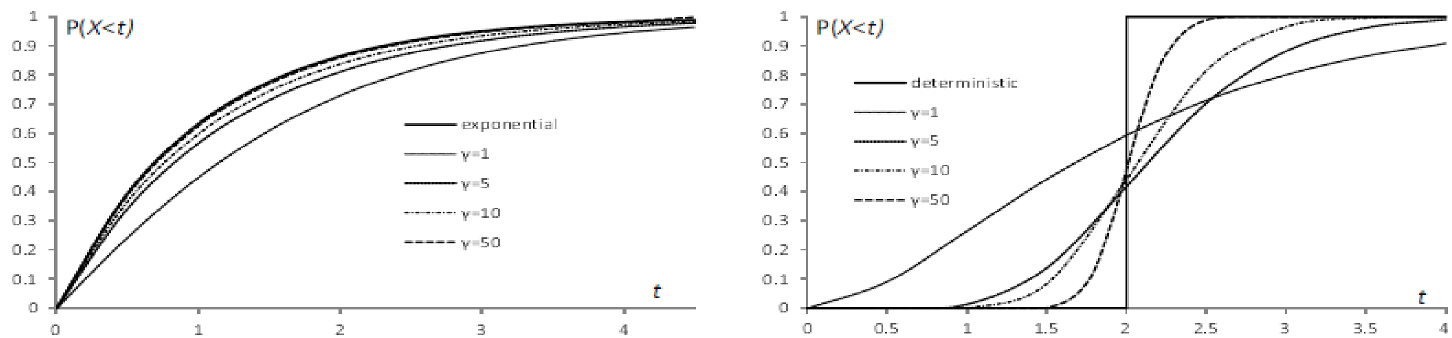
Table 1 Parameters of X_γ to Fit Classical Distributions.

Distribution	Parameters for X_γ
1. Infinite patience	$b = 1,\ r_i = 1$ for $i > 0$
2. Infinite impatience	$b = 0$
3. Exponential (β)	$b = 1$ and $r_i = \frac{\gamma}{\gamma+\beta}$ for $i > 0$
4. Deterministic (τ)	$b = 1,\ r_i = 1$ for $0 < i \le n$ and $r_i = 0$ for $i > n$, where $n \in \mathbb{N}$, and $\gamma = \frac{n}{\tau}$
5. Erlang (N, β)	$b = 1$ and $r_i = \frac{\gamma}{\gamma+\beta} \cdot \frac{\sum_{n=0}^{N-1} \binom{i}{n} \left(\frac{\beta}{\gamma}\right)^n}{\sum_{n=0}^{N-1} \binom{i-1}{n} \left(\frac{\beta}{\gamma}\right)^{n-1}}$ for $i > 0$

Remarks:

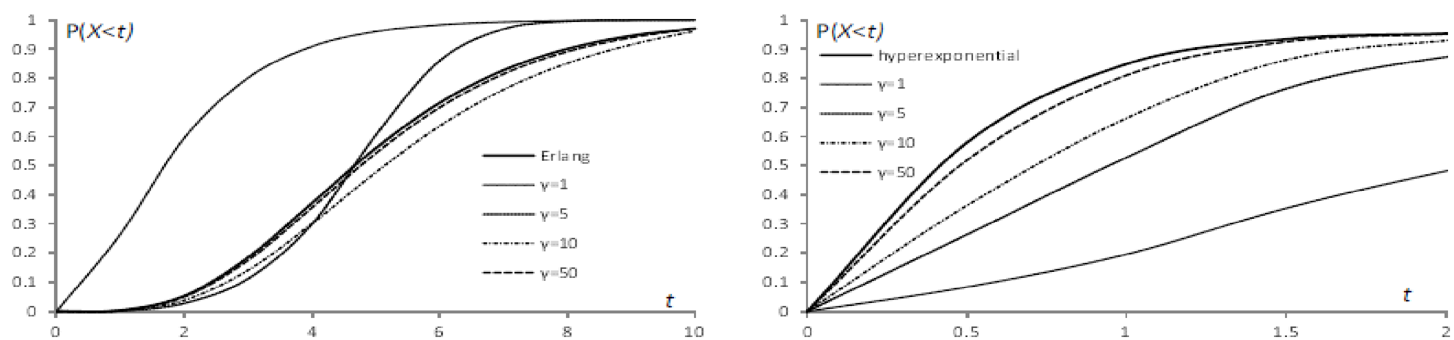
1. Practical difficulty to determine the parameters
2. Weak convergence result

PROPOSITION 1. *With $b = 1$ and $r_i = \frac{\gamma}{\gamma+\beta}$ for $i > 0$, X_γ does not converge in probability to an exponential random variable with parameter β.*



(a) Exponential Distribution with rate 1.

(b) Deterministic distribition with parameter 2.



(c) Erlang distribution with 5 phases and rate 1.

(d) Hyperexponential distribution with parameters $(1, 10\%), (10, 90\%)$.

Figure 2 Convergence of $X_{\gamma,D}$.

Main Result: Transition probabilities

THEOREM 2. *We have*

$$p_{i,i-h} = \begin{cases} \prod_{k=1}^{i} q_k \ \ for\ i = h, 0 < i \le D, \\ (1 - q_{i-h}) \prod_{k=i-h+1}^{i} q_k, \ \ for\ 0 \le h < i \le D, \end{cases} \tag{1}$$

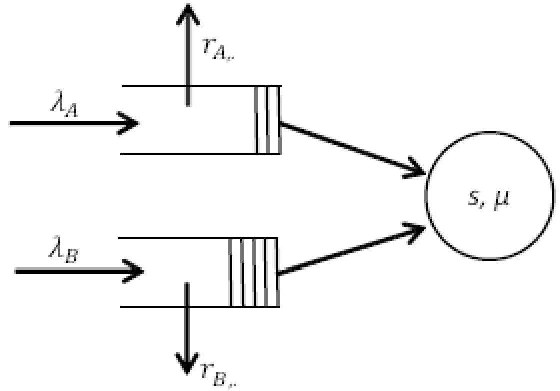
where

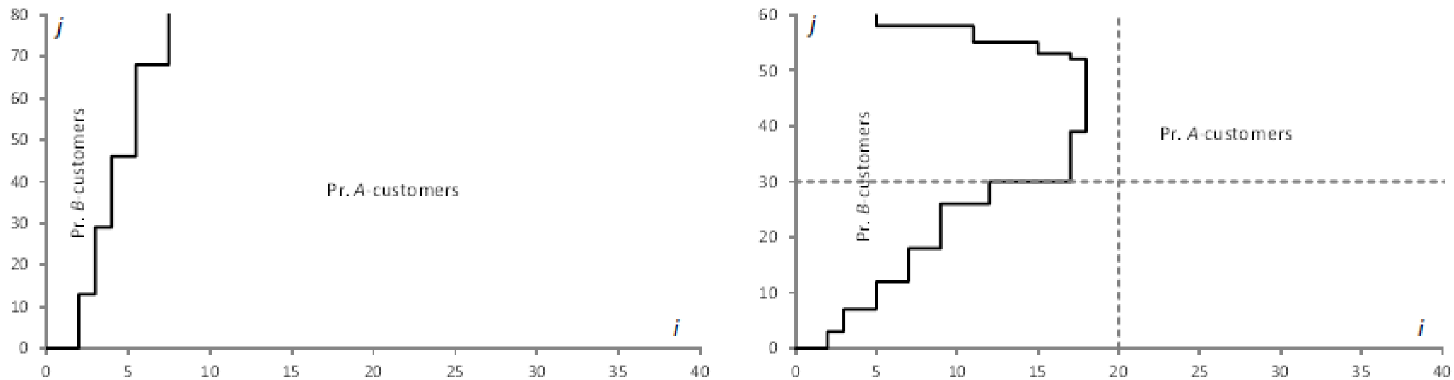
$$q_k = \left[1 + \frac{b\lambda}{\gamma} \prod_{j=1}^{k} r_j \right]^{-1}, \ \ for\ 0 < k \le D. \tag{2}$$

Applicability of the method :

1. Routing optimization:

In the V-design, which customer should be served to minimize the proportion of abandonment?





(a) $c_A(i) = 5i/\gamma$, $c_B(j) = j/\gamma$, $i, j > 0$.

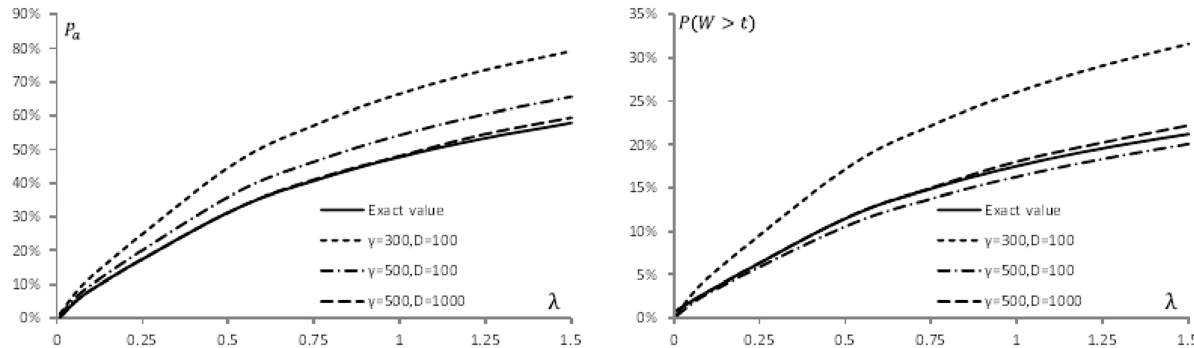
(b) $c_A(i) = 5$ for $i > 20$ and 0 otherwise, $c_B(j) = 1$, for $j > 30$ and 0 otherwise.

Figure 4 Optimal Policies ($\lambda_A = \lambda_B = 5$, $\mu = 1$, $s = 11$, $u_A = 0.1$, $\alpha_{A,1} = 1$, $\alpha_{A,2} = 5$, $u_B = 0.3$, $\alpha_{B,1} = 2$, $\alpha_{B,2} = 3$, $\gamma = 30$, $D = 120$).

Applicability of the method :

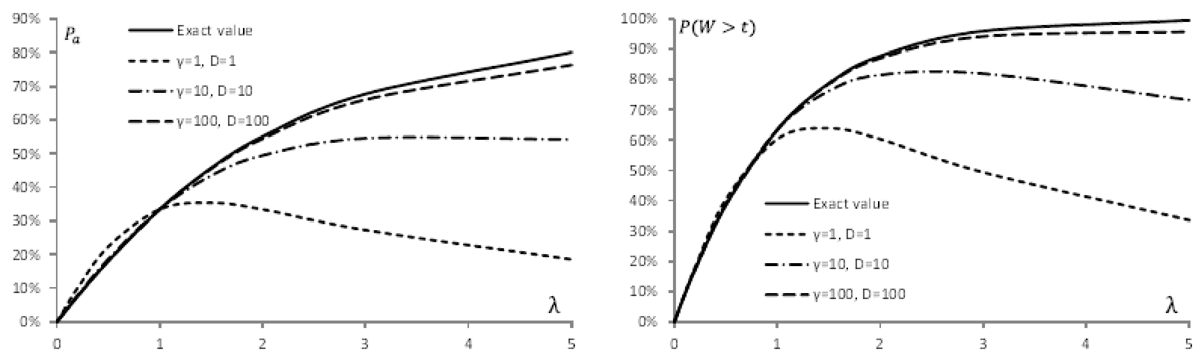
2. Performance Evaluation (using the embedded Markov chain at service initiation instants):

The M/M/s+GI Queue



(a) Proportion of Abandonment. (b) $P(W > t)$.

Figure 5 Convergence of the Performance Measures in the M/M/s+M queue ($s = 1$, $\mu = 1$, $t = 0.1$, $\beta = 10$).



(a) Proportion of Abandonment. (b) $P(W > t)$.

Figure 6 Convergence of the Performance Measures in the M/M/s+D Queue ($s = 1$, $\mu = 1$, $t = 0.1$, $\tau = 1$).

Limitations and Research questions

Limitations:

1. **Homogeneous Poisson arrival process**
2. **FCFS policy**
3. **Numerical method; difficulty to prove the propagation of monotonicity properties (exceptions with deterministic abandonment)**
4. **Decisions to enter service/routing to another system**

Research Questions:

5. **Framework to compare routing based on the quantity versus the experienced waiting time? Under which condition one state definition is better than the other.**
6. **Adaptation of the transition probabilities to non-FCFS discipline**
7. **Propagation of monotonicity properties.**